BULLETIN

OF THE

AMERICAN MATHEMATICAL SOCIETY

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VOLUME XLI, NUMBER 10

OCTOBER, 1935

PUBLISHED BY THE SOCIETY

MENASHA, WIS., AND NEW YORK

1935

Whole No. 433 $9.00 a Year

PUBLISHED MONTHLY

Entered as second class matter April 24, 1926 at the post office at Menasha, Wis., under the act of August 24, 1912. Acceptance for mailing at the special rate of postage provided for in the act of February 28, 1925, embodied in paragraph 4, section 538, P. L. and R., authorized May 9, 1935.

PERIODICAL PUBLICATIONS
OF THE
AMERICAN MATHEMATICAL SOCIETY

BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY

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AMERICAN MATHEMATICAL SOCIETY

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TOPICS IN THE FUNCTIONAL CALCULUS†

BY L. M. GRAVES

PART I. THE THEORY OF FUNCTIONALS

In this lecture it is proposed to outline an abstract theory of functionals, with a development paralleling that of the theory of functions of real variables, and including also a chapter on analytic functionals. In Part II, some applications of the general theory to various sorts of equations are indicated.

From the abstract point of view, the functional calculus is a form of general analysis, and as such it was effectually initiated by Fréchet's thesis in 1906. Since then a large number of researches have been concerned with the topological properties of abstract sets, and with the properties of continuous or semi-continuous functionals.

The postulational basis for an abstract topological theory may take various forms. A general basis consists of a general, that is, unrestricted class $\mathfrak{X}$ of elements x, and an unrestricted function K on $\mathfrak{S}$ to $\mathfrak{S}$, where $\mathfrak{S}$ is the class of all subsets E of $\mathfrak{X}$. The function K is then a set-valued function of sets. The system $(\mathfrak{X}, K)$ constitutes a topological space. It has been shown by Chittenden [7, pp. 294–298]‡ that a related set-function H may always be defined such that the space $(\mathfrak{X}, H)$ has the three properties:

I. $H(D+E)=H(D)+H(E)$.

II. For every set E, $H(E)$ contains $H(H(E))$.

III. If E is finite, $H(E)$ is null.

Such a space $(\mathfrak{X}, H)$ is called an *accessible space* by Fréchet.

If the points of $H(E)$ are called the *points of accumulation* of the set E, then closed sets may be defined as usual. A point of a set E is interior to E in case it is not a point of accumulation of the complement of E. Open sets are those consisting only of interior points. The neighborhoods of a point x may be defined as those sets having x as an interior point. A set E is called *com-*

† An address delivered by invitation of the program committee at the Chicago meeting of this Society, April 19, 1935.

‡ References in brackets are to the bibliography at the end of the paper.

pact in case every infinite subset A of E has at least one point of accumulation, that is, $H(A)$ is not null.

Another basis for a topological theory consists of the system $(\mathfrak{X}, \mathfrak{S}_0)$, where $\mathfrak{S}_0$ is a subclass of the class $\mathfrak{S}$ of all subsets E of $\mathfrak{X}$. If the sets E_0 in the class $\mathfrak{S}_0$ are thought of as the *open* subsets of $\mathfrak{X}$, we are led to definitions of point of accumulation and of derived set $E'=K(E)$ which give us a system $(\mathfrak{X}, K)$. Sierpiński [**29**, pp. 1, 28] has given five postulates for the system $(\mathfrak{X}, \mathfrak{S}_0)$ which are equivalent to the properties I, II, and III for the corresponding system $(\mathfrak{X}, K)$. Hausdorff [**17**, p. 213] considered a system $(\mathfrak{X}, \mathfrak{S}_0)$ in which the sets E_0 in the class $\mathfrak{S}_0$ are related as *neighborhoods* to the points of $\mathfrak{X}$. Hausdorff's set of four postulates is more restrictive than the set given above.

Still another basis for a topological theory is the system (class (L) of Fréchet) $(\mathfrak{X}, \Sigma_c, L)$, where Σ_c is a class of sequences of points of $\mathfrak{X}$, called *convergent sequences*, and L is a function on Σ_c to $\mathfrak{X}$, assigning to each convergent sequence its *limit* [Fréchet, **10**, p. 164]. Fréchet assumed:

(a) If $x_n=x$ for every n, then $L\,x_n=x$.

(b) If $L\,x_n=x$, then every subsequence converges and has the same limit.

In such a system derived sets $E'=K(E)$ are readily defined, yielding a topological space $(\mathfrak{X}, K)$. But it is possible for a compact sequence of distinct points (x_n) to have only one point of accumulation x without being convergent. However, this inconvenience may always be removed by extending the range Σ_c of definition of the function L to a maximum without changing the topological character of the space, that is, without changing the function K [Urysohn, **30**]. If in addition derived sets are always closed, the space $(\mathfrak{X}, K)$ obtained from $(\mathfrak{X}, \Sigma_c, L)$ is an accessible space. If this additional condition does not hold, a related function $H(E)$ may be defined by the process of Chittenden so that the resulting space $(\mathfrak{X}, H)$ is accessible. But a point of accumulation of a set E in the space $(\mathfrak{X}, H)$ need not be the limit of any convergent sequence of points from E. For example, let $\mathfrak{X}$ be the space of all real-valued functions $x(t)$ defined on $0\leqq t\leqq 1$, and define $L\,x_n=x$ to mean ordinary convergence for each value of t. Then the related accessible space $(\mathfrak{X}, H)$ defined by the process of Chittenden (which in this case is also a Hausdorff space) is such that the derived set $H(E)$ of

a given set E contains all functions obtainable from those of E by *repetitions* of the limiting process. Thus if E is the class of continuous functions, $H(E)$ consists of all the Baire functions.

The notion of limit of a sequence may be defined in any accessible space $(\mathfrak{X}, H)$ thus: $L\, x_n = x$ in case every neighborhood of x contains all the points of the sequence from a certain place on. In the example just mentioned, this definition coincides with the original definition of limit. However, in a general accessible space, a sequence may have more than one limit. In a Hausdorff space on the other hand, a sequence can have at most one limit. But it is still possible for a set E to have a point of accumulation which is limit of no sequence of points from E.

In discussing functions defined on abstract spaces it seems convenient to refer to a *numerically-valued* function defined on an abstract space as an *operation*, and to use the word *transformation* to refer to general functional relations between two or more such spaces.

A function or transformation $f(x)$ which transforms an accessible space $\mathfrak{X}$ into another accessible space $\mathfrak{Y}$ is defined to be *continuous* at a point x of $\mathfrak{X}$ in case for every open set B containing $f(x)$ there exists an open set A containing x such that the transform $f(A)$ is contained in B. It follows that a continuous function of a continuous function is continuous. Also a continuous transform of a connected set is connected. If the space $\mathfrak{X}$ is compact and an operation f is continuous on $\mathfrak{X}$, then f is bounded on $\mathfrak{X}$. If every operation f continuous on $\mathfrak{X}$ is bounded on $\mathfrak{X}$, then for each such f the set of its functional values is closed. As a corollary we may say that a real-valued operation f continuous on a closed compact set E has a maximum and a minimum on E. A Borel theorem may be stated for an accessible space as follows. Let the set E be compact and closed, and let $\mathfrak{F}$ be a denumerable family of open sets O covering E, in the sense that each point of E is interior to a set O of the family $\mathfrak{F}$. Then there exists a finite subfamily of $\mathfrak{F}$ which also covers E [see Hildebrandt, **20**, pp. 470, 471; Chittenden, **7**, p. 300].

A metric space $(\mathfrak{X}, \rho)$ consists of a set $\mathfrak{X}$ of elements x and a real-valued operation ρ defined on $\mathfrak{X}\mathfrak{X}$, with the properties:

(1) $\rho(x_1, x_2) = 0$ if and only if $x_1 = x_2$.

(2) $\rho(x_1, x_3) \leqq \rho(x_1, x_2) + \rho(x_3, x_2)$.

It follows from these two properties that the operation ρ is

symmetric and non-negative. If we agree that $L\ x_n = x$ when $\lim \rho(x_n, x) = 0$, we see that a metric space is a class ($\mathfrak{L}$) of Fréchet. It is also a Hausdorff space. It is plain that the distance function may be altered without changing the topological character of the space. For example, we may obtain a new distance function ρ^* by setting $\rho^* = \rho/(1+\rho)$. A fundamental problem is that of determining the conditions under which a topological space $(\mathfrak{X}, K)$ is metrizable. That is, when does there exist a metric ρ defining the same points of accumulation for a set as the relation K? For certain types of topological spaces the problem has been solved by Alexandroff and Urysohn, and by Chittenden [6].

In case the Cauchy condition, $\lim_{m,n} \rho(x_m, x_n) = 0$, is sufficient to ensure the convergence of a sequence (x_n) to a point x of a metric space $(\mathfrak{X}, \rho)$, the space is called *complete*. The property of completeness may be lost by a change of metric, so that it is not a topological property. However, we may consider the topological property of "completeness for some metric ρ." Fréchet prefers to apply the word "complete" to any topological space having this latter property. Likewise, from the topological point of view, a space which is metrizable is called metric. In the sequel we shall be interested in many non-topological properties, so that we shall use the term "metric space" to refer to a space $(\mathfrak{X}, \rho)$ with a definite associated metric ρ, and the term "complete" will be used only with reference to that metric, and not in the Fréchet sense.†

In a metrizable space the Borel theorem holds without the restriction of denumerability of the original family $\mathfrak{F}$ covering the set. Moreover, a continuous transformation f of a closed compact set in a metric space into a metric space is always uniformly continuous.

We have thus far surveyed the elementary ideas and theorems of point-set theory and the fundamental properties of continuous functions as they appear in the foundation of the abstract functional calculus or general analysis. Much of the theory of abstract sets has been passed by in order to adhere to the plan of outlining the analogies of the abstract functional calculus

† Von Neumann has recently given an interesting discussion of the notion of completeness for linear spaces which may not be metrizable. See Transactions of this Society, vol. 37 (1935), pp. 1–20.

with the elementary theories of functions of real and complex variables. We shall turn next to the consideration of vector spaces, and then develop the differential calculus of abstract functions.

A system $(\mathfrak{X}=[x], \mathfrak{A}=[a], +, \cdot, \| \ \|)$ which has the following properties will be called a *normed linear space.*

1. $\mathfrak{A}$ is the real number system or the complex number system.

2. $+$ is an associative transformation on $\mathfrak{X}\mathfrak{X}$ to $\mathfrak{X}$.

3. $\cdot$ is a transformation on $\mathfrak{A}\mathfrak{X}$ to $\mathfrak{X}$, associative with multiplication of numbers in $\mathfrak{A}$.

4. $\cdot$ is distributive with respect to $+$ and with respect to addition of numbers in $\mathfrak{A}$.

5. $1 \cdot x = x$ for every x.

6. $\| \ \|$ is an operation on $\mathfrak{X}$ to $\mathfrak{A}$.

7. $\|x_1+x_2\| \leqq \|x_1\|+\|x_2\|$.

8. $\|a \cdot x\| = |a| \, \|x\|$.

9. There is a unique point Θ in $\mathfrak{X}$ such that $\|\Theta\| = 0$.

A normed linear space is obviously a metric space if we set $\rho(x_1, x_2) = \|x_1 - x_2\|$. In case such a space is also complete, it has been called by Fréchet "un espace (D) vectoriel complet" or a Banach space. Fréchet prefers to distinguish between the "points" of space and the "vectors" associated with ordered pairs of points [**8**; **10**, pp. 123–146, 201–204]. Thus an "affine topological space" consists of a system $(\mathfrak{X}, K)$ with an associated system of vectors $(\mathfrak{Q}=[\xi], +, \cdot, \| \ \|)$ having certain properties. For most purposes I can see no gain in this distinction, since by selecting arbitrarily an "origin" Θ in the space $\mathfrak{X}$ a correspondence may be set up between the points x and vectors ξ. Omitting the distinction simplifies the notation and shortens the list of postulates.

In this connection it should be noted that in certain interesting linear spaces $(\mathfrak{X}, K)$ which are also metrizable, no metric can be defined in terms of a norm satisfying Postulate 8 along with all the others [Fréchet, **8**, p. 50]. Two of these are the space $\mathfrak{M}$ of all measurable functions with limit defined in terms of convergence in measure or approximate convergence, and the space (E_ω) of Fréchet consisting of all infinite sequences of numbers, that is, points in infinitely many dimensions, with

limit defined in terms of convergence of each coordinate. For this reason Fréchet and Banach have considered vector spaces of a more general type than the normed linear spaces, and have derived a few properties of linear and polynomial transformations of such spaces [9; 2, pp. 20–52]. However, this intermediate domain seems not to have aroused much interest.

A transformation $G(x)$ of a normed linear space $\mathfrak{X}$ into a normed linear space $\mathfrak{Y}$ is called *linear* in case

$$G(a_1x_1 + a_2x_2) = a_1G(x_1) + a_2G(x_2)$$

for every pair of points x_1, x_2 and every pair of numbers a_1, a_2. In case such a transformation is continuous, it is also bounded or modular, that is, there is a number M such that $\|G(x)\| \leqq M\|x\|$ for every x. The greatest lower bound of effective M's is called the *bound* or *modulus* or *norm* of G and may be denoted by $M(G)$.

An additional concept which has been found useful in connection with linear spaces is that of *weak convergence*, as distinguished from convergence in terms of the norm. Schauder has proposed the following list of postulates to characterize the concept [27, p. 664]. Let $\mathfrak{X}$ be a normed linear space, and let wL be a single-valued function defined on a class Σ_w of sequences of elements of $\mathfrak{X}$, and with functional values in $\mathfrak{X}$. The sequences of Σ_w are called *weakly convergent*. The properties assumed are:

(a) If $\lim \|x_n - x\| = 0$, then $wL\ x_n = x$.

(b) If $wL\ x_n = x$, $wL\ x_n' = x'$, $\lim a_n = a$, $\lim a_n' = a'$, then $wL\ (a_nx_n + a_n'x_n') = ax + a'x'$.

(c) If a sequence (x_n) is weakly convergent, the sequence of norms $(\|x_n\|)$ is bounded.

(d) If $wL\ x_n = x$ and $\|x_n\| \leqq M$, then $\|x\| \leqq M$.

(e) Every subsequence of a weakly convergent sequence has the same weak limit.

Such a system $(\mathfrak{X}, \Sigma_w, wL)$ is plainly a class $(\mathfrak{L})$ of Fréchet.

In a given space, various definitions of weak convergence having the specified properties are usually possible. For example, weak convergence may always be identified with convergence in the norm. In this case the concept would be sterile. An important property which may sometimes be obtained by a proper choice of the definition is: (C_w) every bounded set is

weakly compact, that is, every sequence which is bounded in the norm has a weakly convergent subsequence.

In case we define weak convergence as follows: $wL\ x_n = x$ in case $\lim f(x_n) = f(x)$ for every continuous linear operation f defined on $\mathfrak{X}$, it may be shown that the weak limit x of a sequence (x_n) belongs to the extension of the set $[x_n]$ to be linear and closed [Banach, 2, p. 134]. Also every continuous linear transformation G is weakly continuous, that is, $wL\ G(x_n) = G(x)$ whenever $wL\ x_n = x$ [2, p. 143].

A functional transformation G of a set X_0 into part or all of a space $\mathfrak{Y}$ is commonly called *completely continuous* in case it is continuous and transforms every bounded subset E of X_0 into a compact set $G(E)$. Let us say that G is *totally continuous* in case it transforms every sequence which is weakly convergent according to the definition just given into a sequence convergent according to the norm. It may be shown by examples that neither of these properties implies the other. However, if every bounded subset E of X_0 is weakly compact, then a totally continuous transformation G of X_0 is also completely continuous. On the other hand a completely continuous *linear* transformation G is also totally continuous [Banach, 2, p. 143].

It may not be out of place to remark here that the spaces of "modular" functions considered by E. H. Moore in his second general analysis theory are all normed linear spaces. Professor Moore defined a sequence of modular functions ξ_n to be convergent "in mode one" in case the norms are bounded, and the sequence $\xi_n(p)$ converges for each p. He showed that convergence in mode one is equivalent to weak convergence as defined above, and that every bounded set of modular functions is always weakly compact.

An important theorem on sets of functional operations is the following. Let the sequence of operations $g_n(x)$ defined on the complete normed linear space $\mathfrak{X}$ have the following properties:

(1) $|g_n(a_1x_1+a_2x_2)| \leqq |a_1|\,|g_n(x_1)| + |a_2|\,|g_n(x_2)|$.

(2) For every n, $|g_n(x)|$ is continuous in x.

(3) For every x, $g_n(x)$ is bounded with respect to n.

Then the operations $|g_n(x)|$ are continuous in x uniformly with respect to n. A more general form of this theorem was proved by Hildebrandt† [19]. An immediate consequence of the theorem is

† The general form of the theorem follows readily by an indirect proof

the proposition that the limit $f(x)$ of a sequence of linear continuous transformations $f_n(x)$ convergent for every point of the space $\mathfrak{X}$ is also continuous. Another consequence is the proposition that a multilinear transformation which is continuous in each of its arguments separately is continuous in all its arguments together.

There are several conceptions of differentiation which are useful in the functional calculus. Perhaps the first to be used was the very general notion of a variation, which gave to the calculus of variations its name. This notion has passed through several stages, and we shall notice only one modern form of it. Let f be a functional transformation defined on an open set X_0 in a normed linear space $\mathfrak{X}$, with functional values in another such space $\mathfrak{Y}$. Suppose that

(D_1) for every element δx in $\mathfrak{X}$ there exists the derivative

$$\frac{d}{d\alpha} f(x + \alpha\delta x)\Big|^{\alpha=0} \equiv \delta f(x, \delta x).$$

This function $\delta f(x, \delta x)$ is then called the *variation* of the function f at the point x.

Note that we have used here the notion of the derivative of a function $g(\alpha)$ of a numerical variable whose functional values lie in a normed linear space. The definition of such a derivative is exactly analogous to the one used for numerically-valued functions and it has similar properties. A like statement is true concerning the Riemann integral of such a function $g(\alpha)$ [see Graves, **13**], and we shall have several occasions to use such integrals. Various definitions have also been given for the corresponding generalization of the Lebesgue integral, but we shall not need them.

It may readily be shown that the variation $\delta f(x, \delta x)$ of a functional transformation $f(x)$ as defined by statement (D_1) always has the property that it is homogeneous of the first degree in δx. But it may fail to have many of the usual properties asso-

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from the special case stated above. The special case in turn is an easy corollary of the theorem that a complete metric space is of the second category, and the following theorem: If the sequence of non-negative and lower semi-continuous operations $f_n(x)$ is bounded with respect to n for each point x of a set X_0 of the second category, then there is a sphere K in the space $\mathfrak{X}$ on which the operations $f_n(x)$ are bounded uniformly with respect to x and n [Banach, **2**, p. 19].

ciated with derivatives of functions of a single variable, as is shown by the fact that every function $f(x)$ homogeneous of the first degree has a variation at the origin, $\delta f(\Theta, \delta x) = f(\delta x)$. In order to obtain a theory of differentiation with more content, we may suppose that the variation $\delta f(x, \delta x)$ of a functional transformation $f(x)$ has one or more of the following additional properties:

(D_2) $\delta f(x, \delta x)$ is linear in δx.

(D_3) $\delta f(x, \delta x)$ is continuous in δx.

$$(D_4) \quad \lim_{\|\delta x\|=0} \frac{\|f(x+\delta x) - f(x) - \delta f(x, \delta x)\|}{\|\delta x\|} = 0.$$

(D_5') $\delta f(x, \delta x)$ is defined and continuous with respect to x at each point x in X_0 uniformly for $\|\delta x\| = 1$.

In case a property (D) holds for every point x of X_0 we shall denote the corresponding property by (D'). It may be shown that a function $\delta f(x, \delta x)$ having the relation to $f(x)$ expressed in properties (D_2) and (D_4) always has property (D_1) also. Moreover, properties (D_1'), (D_2'), and (D_5') imply (D_4'), provided the space $\mathfrak{Y}$ of functional values is complete. When (D_2), (D_3), and (D_4) are satisfied, we say that the function f has a *total differential* $\delta f(x, \delta x)$ at the point x. The importance of the notion of total differential has been emphasized by Fréchet on various occasions. For example, a function f having a total differential at a point x_0 is always continuous at x_0. Moreover, if also $g(z)$ has a total differential at z_0, and $g(z_0) = x_0$, then the function $f[g(z)]$ has a total differential at z_0, equal to $\delta f[x_0, \delta g(z_0, \delta z)]$. These theorems do not hold for functions having only variations in the weak sense expressed by condition (D_1) alone. However, we shall see later that when the function f is continuous and the derivative in (D_1) is taken with respect to a complex variable α, then property (D_1') implies all the others, provided again that the space $\mathfrak{Y}$ of functional values is complete.

The differentiability properties of a functional operation or transformation $f(x)$ may change when the continuity properties of the space $\mathfrak{X}$ are changed by altering the norm $\|x\|$. For an example we may take a calculus of variations integral

$$f[x] = \int_0^1 \phi(t, x, x')dt,$$

where the space $\mathfrak{X}$ consists of all functions $x(t)$ of class C' and vanishing at $t=0$ and $t=1$, and $\|x\| = \max |x(t)|$. Then

$$\delta f[x, \delta x] = \int_0^1 \{\phi_x \delta x + \phi_{x'} \delta x'\} dt$$

has properties (D_1) and (D_2). Property (D_3) also holds when $x(t)$ is of class C'', but an example shows that it may fail if $x(t)$ is merely of class C'. Property (D_4) fails in general, since the functional operation $f[x]$ is not usually even continuous. However, if we set $\|x\| = \max \{|x(t)|, |x'(t)|\}$, then in our example $\delta f[x, \delta x]$ has all of properties (D_2'), (D_3'), (D_4'), (D_5') in a suitably chosen region X_0, and the operation $f[x]$ is itself continuous.

The variety of useful definitions of differentiation is greater when we consider variations of higher order. There is time to mention only a few. We shall say that a transformation $f(x)$ has an nth variation $\delta^n f(x, \delta x)$ at a point x in case for every δx there exists the nth derivative

$$\frac{d^n}{d\alpha^n} f(x + \alpha \delta x)\Big|^{\alpha=0} \equiv \delta^n f(x, \delta x).$$

The nth variation $\delta^n f(x, \delta x)$ is always homogeneous of degree n in δx. Moreover, if we set

$$R_{n+1}(x, \delta x) = f(x + \delta x) - f(x) - \delta f(x; \delta x) - \cdots - \frac{1}{n!} \delta^n f(x; \delta x),$$

we have

$$\lim_{\alpha=0} \frac{\|R_{n+1}(x, \alpha \delta x)\|}{\alpha^n} = 0.$$

There is valid also a generalization of Taylor's theorem [Graves, **13**, p. 173], namely, if $f(x)$ is defined and has an nth variation at each point of a convex region X_0, and if the space $\mathfrak{Y}$ in which the functional values lie is complete, and if x and $x+\delta x$ lie in X_0, and $\delta^n f(x+t\delta x, \delta x)$ is a continuous function of t, then

$$R_n(x, \delta x) = \int_0^1 \delta^n f(x + t\delta x, \delta x)(1 - t)^{n-1} dt/(n - 1)!.$$

We might also say that a transformation $f(x)$ has a second variation at x_0 in case f has a first variation $\delta f(x, \delta x)$ for each x near x_0, and $\delta f(x, \delta x)$ has a first variation $\delta^2 f(x_0, \delta x, \delta_1 x)$ at x_0 for every δx. The nth variation would then be defined by induction, and would involve n independent variations $\delta_1 x, \cdots, \delta_n x$. In case $\delta f(x, \delta x)$ has the properties (D_2'), (D_3'), and (D_4') of a total differential, and $\delta^2 f(x_0, \delta x, \delta_1 x)$ has the properties (D_3), (D_3), and (D_4) for each δx, according to Fréchet $\delta^2 f(x_0, \delta x, \delta_1 x)$ is called the *second differential* of f at x_0. In this case it may be shown that $\delta^2 f(x_0, \delta x, \delta_1 x)$ is a symmetric function of δx, $\delta_1 x$, and hence that it is linear and continuous in δx and in $\delta_1 x$. When the space $\mathfrak{X}$ is complete, it follows that $\delta^2 f(x_0, \delta x, \delta_1 x)$ is continuous in $\delta x, \delta_1 x$ together.

A function $f(x_1, \cdots, x_k)$ of a finite number of real variables is commonly said to be of class $C^{(n)}$ on an open region in the x-space when it has continuous partial derivatives on that region up to and including those of the nth order. We may generalize this notion to our abstract situation as follows [Hildebrandt and Graves, **18**, pp. 135–144]. We say that $f(x)$ is of class C' on an open region X_0 in case f has a total differential $\delta f(x, \delta x)$ at every point of X_0, which also satisfies condition (D_5'). The function f is of class $C^{(n)}$ on X_0 if f is of class C' and $\delta f(x, \delta x)$ is of class $C^{(n-1)}$ uniformly for $\|\delta x\| = 1$. When $n \geqq 3$, this implies, for example, that $\delta^3 f(x, \delta_1 x, \delta_2 x, \delta_3 x)$ is continuous in $\delta_3 x$ uniformly for $\|\delta_1 x\| = \|\delta_2 x\| = 1$. Using this definition it is possible to prove, for example, that if $f(x)$ is of class $C^{(n)}$ on X_0, and if $g(z)$ is of class $C^{(n)}$ on Z_0 and has its functional values in X_0 then the function $h(z) = f[g(z)]$ is of class $C^{(n)}$ on Z_0 [Hildebrandt and Graves, **18**, p. 144]. While the proof of this theorem is rather difficult, the other formal rules of the differential calculus which have meaning in this general situation are readily derived.

In case $\mathfrak{X}$ and $\mathfrak{Y}$ are complete normed linear spaces with the complex number system $\mathfrak{A}$ as the associated number system, we may develop an interesting theory of analytic functional transformations f on a region X_0 to $\mathfrak{Y}$. The essential features of this were indicated in certain special cases by Gateaux [**11**, **12**]. Let us consider first the case of a function $f(z)$ on A_0 to $\mathfrak{Y}$, where A_0 is an open set in the complex plane. The definition of derivative of $f(z)$ and of ordinary point for $f(z)$ have the usual

form. A function $f(z)$ is said to be *holomorphic* on a set S in case every point of S is an ordinary point for $f(z)$. Line integrals of such functions have the usual elementary properties, and Cauchy's theorem and integral formula hold true. A Laurent expansion for $f(z)$ about an isolated singularity may be derived in the usual way, and the notion of analytic continuation also generalizes. All this was indicated by Wiener [31].

Let us turn now to the general case of a functional transformation $f(x)$ defined on an open set X_0 in a normed linear space. We say that a point x_0 is an ordinary point for f in case f is continuous and has a first variation at each point x near x_0. If each point of X_0 is an ordinary point for $f(x)$, then f is *holomorphic* on X_0.

A transformation $f(x)$ on $\mathfrak{X}$ to $\mathfrak{Y}$ is called a *polynomial transformation* in case $f(x_1+\lambda x_2)$ is a polynomial in the numerical variable λ with coefficients in the space $\mathfrak{Y}$, for every x_1 and x_2 in $\mathfrak{X}$. This definition reduces to the ordinary meaning of the word polynomial in case f is a numerically-valued function of a finite number of numerical variables. A continuous and homogeneous polynomial transformation $P_n(x)$ of degree n may be polarized to give a multilinear form $Q(x_1, x_2, \cdots, x_n)$ linear and continuous in each of its arguments, such that the relation $Q(x, \cdots, x) = P_n(x)$ holds. The kth differential of $P_n(x)$ is then $Q(\delta_1 x, \cdots, \delta_k x, x, \cdots, x)$ if $k \leqq n$, and vanishes if $k > n$.

Let the transformation $f(x)$ be defined by a series of the form

$$f(x) = \sum_0^\infty P_n(x). \tag{1}$$

There is a theorem on the nature of the region of convergence of such a series, analogous to Abel's theorem for ordinary power series. Let X_0 denote the open set remaining after discarding the boundary points of the set on which the series (1) converges. Then if the point x is in X_0 and $|\lambda| \leqq 1$, the point λx is also in X_0. We shall call a region X_0 having this property a *region of circular type* about the origin. In order to obtain interesting properties, we assume that the series (1) converges uniformly on sets E which are compact on X_0. An equivalent condition states that every point x_0 of the region X_0 has a neighborhood on which the series converges uniformly. It follows that the sum

$f(x)$ of the series is holomorphic on X_0, and in fact of class $C^{(\infty)}$.

Conversely, we have also a Taylor's theorem. That is, if a transformation $f(x)$ is holomorphic on X_0, then about each point x_0 in the region X_0 the transformation $f(x)$ is expansible in a series

$$f(x) = \sum_0^\infty P_n(x_0;\ x - x_0) \tag{2}$$

of continuous homogeneous polynomial functions of $(x-x_0)$, converging uniformly on a neighborhood of x_0, so that $f(x)$ is necessarily of class $C^{(\infty)}$. In case X_0 is a region of circular type about the point x_0, the Taylor expansion (2) for $f(x)$ converges uniformly on sets E compact on X_0. Also $\|f(x)\|$, being continuous, is bounded by a constant M for $\|x-x_0\| \leqq \rho$, where ρ is sufficiently small, so that by Cauchy's integral formula,

$$\|P_n(x_0;\ x - x_0)\| \leqq M(\|x - x_0\|/\rho)^n.$$

The sum of a series of functional transformations, $f(x) = \sum f_n(x)$, where each $f_n(x)$ is holomorphic on X_0, is also holomorphic on X_0, provided the series converges uniformly on sets E compact on X_0. Moreover, if f on X_0 to $\mathfrak{Y}$ is holomorphic on X_0 and g on U_0 to X_0 is holomorphic on U_0, then the transformation $h(u) = f[g(u)]$ is holomorphic on U_0. If $f(x, u)$ is a function of two points x and u which is continuous in x and u together and holomorphic with respect to x and u separately for x in X_0 and u in U_0, then $f(x, u)$ is a holomorphic function of the composite variable (x, u), and conversely. Finally, we may make use of the process of analytic continuation. However, we do not expect to secure much information about the nature of the singularities of an analytic functional transformation $f(x)$.

Part II. Implicit Function Theorems and Applications

We may apply the preceding theory to study the solutions of various sorts of equations. Thus it may be desired to solve the equations

$$f_i(x_1, \cdots, x_m, y_1, \cdots, y_n) = 0, \qquad (i = 1, \cdots, n),$$

for the y's as functions of the x's. This system of equations may be written $f(x, y) = 0$, where the range of x is a region of the m-dimensional x-space, the range of y is a region of the n-dimensional y-space, and the functional values of f are in the y-space. The differential equation $f(x, y, dy/dx) = 0$ has for the unknown a function $y = y(x)$ in the space of functions having continuous derivatives. The dependence of the solution $y(x)$ on the function f may be investigated by the methods of the functional calculus [Bliss, 4]. From our general point of view, algebraic equations, differential equations, integral equations, and many other types are all gathered under one head. However, we may still wish to consider a variety of hypotheses, and there are different methods available.

The simplest method is that of successive approximations. Let $(\mathfrak{Y}, \rho)$ be a complete metric space, and let $(y_0)_a$ denote the neighborhood of y_0 consisting of those points y for which $\rho(y, y_0) < a$. Consider an equation of the form

$$y = F(y), \tag{3}$$

where the transformation F is a function on $(y_0)_a$ to $\mathfrak{Y}$ with the properties:

(a) It satisfies a Lipschitz condition with constant $k < 1$, that is, $\rho(F(y_1), F(y_2)) \leqq k\rho(y_1, y_2)$.

(b) $\rho(F(y_0), y_0) < (1-k)a$.

Then the method of successive approximations shows at once that there is a unique solution of equation (3) in the neighborhood $(y_0)_a$; that is, there is a unique *fixed point* of the transformation $\bar{y} = F(y)$. In case the transformation F depends on a parameter x, which may be a point in an accessible or a metric space, the solution of equation (3) becomes a function $y = Y(x)$, and the theorem is properly called an implicit function theorem. If the function $F(x, y)$ is continuous in its two arguments, then the solution $y = Y(x)$ will also be continuous [Hildebrandt and Graves, 18, pp. 133–135].

Suppose we wish to consider an equation of the form $G(y) = \Theta$, where G is a transformation defined on a neighborhood $(y_0)_a$ with functional values in a space $\mathfrak{Z}$, and $\mathfrak{Y}$ and $\mathfrak{Z}$ are complete normed linear spaces. We may apply a generalization of Newton's method to obtain a solution (a process which is equivalent to reducing the equation to the form (3) and applying the

method of successive approximations), provided: (a) the point y_0 is a sufficiently close approximation to a solution of the equation, (b) the function G has a differential $\delta G(y_0, \delta y)$ at y_0, and (c) this linear transformation $\delta G(y_0, \delta y)$ on $\mathfrak{Y}$ to $\mathfrak{Z}$ has an inverse.

The question of the existence and properties of an inverse for a linear transformation G has been the subject of extensive investigations. I shall mention only a few elementary theorems. The first states that if a continuous linear transformation G has an inverse H, then H is also linear and continuous [Banach, **2**, p. 41]. The second states that if the space $\mathfrak{Y}$ is complete, and if the continuous linear transformation G of $\mathfrak{Y}$ into all or part of itself has norm $M(G)$ less than unity, then the transformation $K(y) = y - G(y)$ always has an inverse which may be expressed in terms of the usual series of iterates of G [Banach, **1**, p. 161; Hildebrandt and Graves, **18**, p. 145]. The third may be stated as follows. Let the space $\mathfrak{Y}$ be complete, and let $G(y, w)$ be a continuous linear transformation of $\mathfrak{Y}$ into $\mathfrak{Z}$ which depends also on a parameter w ranging over an accessible space. Suppose that G is continuous in w at $w = w_0$ uniformly for $\|y\| = 1$, and that $G(y, w_0)$ has an inverse. Then for w in a sufficiently restricted neighborhood of w_0, $G(y, w)$ has an inverse $H(z, w)$, and H is continuous at $w = w_0$ uniformly for $\|z\| = 1$ [Hildebrandt and Graves, **18**, p. 146].

A general implicit function theorem may be stated as follows. Let $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{Z}$ be normed linear spaces, of which $\mathfrak{Y}$ and $\mathfrak{Z}$ are also complete. Suppose that $G(x,y)$, with functional values in $\mathfrak{Z}$, is of class $C^{(n)}$ on a region R in the composite space $(\mathfrak{X}, \mathfrak{Y})$, and let (x_0, y_0) be an initial solution of the equation

$$G(x, y) = \Theta \tag{4}$$

at which the partial differential $\delta_y G(x_0, y_0; \delta y)$ has an inverse. Then in a sufficiently small neighborhood $(y_0)_a$ the equation (4) has a unique solution $y = Y(x)$ defined on a neighborhood $(x_0)_b$, and the solution $Y(x)$ is also of class $C^{(n)}$ [Hildebrandt and Graves, **18**, p. 150].

From our discussion of analytic functional transformations it is plain that when the transformation $G(x, y)$ is analytic, the solution $y = Y(x)$ is also analytic.

It is frequently of interest to know how far a solution $y = Y(x)$ of equation (4) may be extended. Let us define a *sheet of points*

in the composite space $(\mathfrak{X}, \mathfrak{Y})$ to be an arcwise connected set S with the additional property that for every point (x_1, y_1) of S there are positive constants a and b such that no two points of S in the neighborhood $(x_1, y_1)_a$ have the same projection x, and every point x in the neighborhood $(x_1)_b$ is the projection of a point (x, y) of S contained in $(x_1, y_1)_a$. Then under the hypotheses of the theorem just stated there is a uniquely determined maximal sheet S of solutions of equation (4) which (a) passes through the initial solution (x_0, y_0); (b) is composed entirely of points (x, y) at which the differential $\delta_y G(x, y; \delta y)$ has an inverse; and (c) has for its boundary points only boundary points of the region R and points at which $\delta_y G(x, y; \delta y)$ ceases to have an inverse. This theorem is readily proved by defining S as the logical sum of all sheets having the properties (a) and (b). It may be noted that a sheet of solutions determines a single-valued function $Y(x)$ in a neighborhood of each of its points, and each of these functions is of class $C^{(n)}$.

Various applications of the general implicit function theorem described above have been made to differential, integral, and integro-differential equations, and to the calculus of variations [Graves, **14**, pp. 535 ff.; **15**; **16**, p. 677]. The theorem is a very powerful one, and simple in application. The principal problem in any special case is to determine when the partial differential $\delta_y G(x, y; \delta y)$ has an inverse. Let us consider a simple application to an ordinary differential equation with boundary conditions,

$$(5) \qquad y'' = h(t, y, y'), \qquad y(a) = A, \qquad y(b) = B.$$

Let $\mathfrak{Y}$ be the space of all real functions $y(t)$ of class C'' on $a \leqq t \leqq b$, with $\|y\|$ defined in the usual way as the maximum of $|y(t)|, |y'(t)|, |y''(t)|$ on the interval. Let $\mathfrak{H}$ be the space of all real functions $h(t, y, y')$ continuous in t and of class $C^{(m)}$ in y and y', for (t, y, y') in a certain bounded region R. Let $\|h\|$ be the maximum of $|\partial^{i+j} h / \partial y^i \partial y'^j|$ for (t, y, y') in R and $i+j = 0, \cdots, m$. Let $\mathfrak{Z}$ be the composite space consisting of all continuous functions $z(t)$ and pairs of numbers A, B, with $\|z, A, B\| =$ maximum of $|z(t)|, |A|, |B|$. Then the equations

$$y'' - h(t, y, y') = z(t), \qquad y(a) = A, \qquad y(b) = B,$$

define a transformation $G(y, h)$ of class $C^{(m)}$ on a region of the composite space $(\mathfrak{Y}, \mathfrak{H})$ to the space $\mathfrak{Z}$. In this special problem

a necessary and sufficient condition that the partial differential $\delta_y G(y, h; \delta y)$ have an inverse transformation is that the equation of variation

$$(6) \qquad \delta y'' - h_{y'}\delta y' - h_y \delta y = 0, \qquad \delta y(a) = 0 = \delta y(b),$$

be incompatible, that is, have only the solution $\delta y \equiv 0$. Hence if an initial solution of equation (5) is known at which (6) is incompatible, this solution may be continued, allowing the function h and the end values A and B to vary until a point is reached which lies on the boundary of the region of admissible points, or at which equation (6) becomes compatible. This solution $y(t) = Y[h, A, B \mid t]$ will be of class $C^{(m)}$ as a function of h, A, and B.

This result may be applied to the calculus of variations as follows. Let $y = y(t)$ be an extremal for an integral

$$\int_a^b f(t, y, y')dt$$

joining two points (a, A) and (b, B). Suppose that

$$(7) \qquad f_y, \quad f_{y't}, \quad f_{y'y}, \quad f_{y'y'}$$

have continuous partial derivatives up to order m with respect to y and y', and let $\|f\|$ be defined as the maximum of the absolute values of the partial derivatives (7) and their partial derivatives up to order m. Suppose that $f_{y'y'} \neq 0$ along the initial extremal, and that the end points of this extremal are not conjugate. Then when A, B, and the integrand f are varied, the extremal joining (a, A) and (b, B) will vary continuously and be of class $C^{(m)}$ as a function of f, A, and B, and the process may be continued until an extremal arc is reached which either is singular, that is, has $f_{y'y'} = 0$ at some point, or else has end points which are conjugate. If the initial extremal is a minimizing extremal, then the varied extremal will continue to furnish at least a weak relative minimum for the integral.

The method described may be applied also to partial differential equations of elliptic type. An essential theorem for that purpose is the one which states that if a linear elliptic partial differential equation of the second order has at most one solution taking given boundary values, then it always has a solution for arbitrary continuous boundary values, at least when the

shape of the boundary is sufficiently restricted. Schauder has recently given a proof of this based on the functional calculus and on the known solvability of the equation

$$(8) \qquad D(u) \equiv u_{xx} + u_{yy} = f(x, y), \qquad (u = \phi \text{ on the boundary}),$$

and known inequalities affecting the solutions [28]. Schauder gives the theorems for the case of n independent variables, but for simplicity they are stated here for the case $n=2$. Inequalities affecting the solutions of the equation

$$(9) \qquad L(u) \equiv a(x, y)u_{xx} + 2b(x, y)u_{xy} + c(x, y)u_{yy} = f(x, y), \qquad (u = \phi \text{ on the boundary}),$$

where $a>0$, $ac-b^2=1$, are first derived. All the functions appearing in (9) are supposed to be α-Hölder continuous, $(0<\alpha<1)$, and the norms $\|u\|$, $\|f\|$, and $\|\phi\|$ involve the Hölder constants of those functions. The coefficients a, b, and c are supposed to be β-Hölder continuous, where $\beta>\alpha$. The inequalities proved yield a type of weak compactness of the set of solutions of the one-parameter family of equations

$$(10) \qquad K(\lambda, u) \equiv \frac{(1-\lambda)D(u) + \lambda L(u)}{P(\lambda)} = f, \qquad (u = \phi \text{ on the boundary}),$$

where $P(\lambda)$ is a certain function which is continuous and not zero for $0 \leqq \lambda \leqq 1$, and $\|f\|$ is bounded. Hence if (10) has a solution $u=u_n$ for $\lambda=\lambda_n$, and λ_n approaches λ, then a subsequence of the sequence u_n is weakly convergent to a solution u which also satisfies the inequalities. Now the transformation $K(\lambda, u)$ is continuous in λ and u and linear in u, so that it will continue to have an inverse for λ near a value λ_0 for which it is known to have an inverse. Since the family (10) reduces to equation (8) for $\lambda=0$ and to equation (9) for $\lambda=1$, and equation (8) always has a unique solution, it follows that equation (9) always has a unique solution. Let the inverse transformation so defined be denoted by $u=T(f)$. Then the general elliptic linear equation

$$(11) \qquad L(u) + d(x, y)u_x + e(x, y)u_y + g(x, y)u = h(x, y), \qquad (u = \phi \text{ on the boundary}),$$

may be transformed into

$$(12)\quad f = h - d\frac{\partial}{\partial x}T(f) - e\frac{\partial}{\partial y}T(f) - gT(f) \equiv h + S(f),$$

and the transformation $S(f)$ turns out to be completely continuous. Thus by use of the Riesz theory of linear transformations of the form (12), where $S(f)$ is completely continuous [Riesz, **24**; Hildebrandt, **21**; Schauder, **26**], it follows that if (11) has at most one solution, then it always has a solution.

The foregoing proofs are made first for the case when the boundary values ϕ are supposed to have continuous second derivatives satisfying a Hölder condition, and the theorems are extended to the case of arbitrary continuous boundary values by use of certain additional inequalities. This proof by Schauder is a good example of the advantages offered by the general methods of the functional calculus. No use is made of an elementary solution or of Green's function.

A number of other fixed point theorems and implicit function theorems have been proved for general spaces by topological methods. We remark that topological methods yield no information about the differentiability of functions defined implicitly.

Throughout the following paragraphs it is understood that the spaces $\mathfrak{X}$ and $\mathfrak{Y}$ are complete normed linear spaces.

A very general fixed point theorem is as follows. Let f be a continuous transformation of a convex closed set S in the space $\mathfrak{X}$ into a part of itself, such that the transformed set $f(S)$ is compact. Then the transformation f leaves one point of S fixed. A fixed point theorem for n-dimensional space was proved by Brouwer. The proof of the general theorem depends on an extension of the Brouwer theorem, and is made by defining transformations f_n which approximate the transformation f on n-dimensional subsets of the set S. The first extension to function spaces was made by Birkhoff and Kellogg in 1922 [**3**]. Theorems for general spaces were proved later by Schauder [**25**].

The following theorem on inversion of functional transformations is readily proved. Let $y = F(x)$ be a transformation defined and continuous on an arcwise connected region X_0 of the space $\mathfrak{X}$, which is locally invertible, that is, the equation $y = F(x)$ has a unique continuous solution near each of its solutions. Suppose also that whenever the sequence $F(x_n)$ converges, the sequence

(x_n) is compact on X_0. Then F is a topological transformation of X_0 into the whole space $\mathfrak{Y}$. It is clear that a fixed point theorem for a transformation $G(x)$ may be obtained as a corollary by setting $F(x)=x-G(x)$. The theorem may be proved as follows. Let $Y_0=F(X_0)$, and let (y_n) be a sequence of points of Y_0 converging to a point y'. Then the corresponding sequence (x_n) is compact on X_0 and hence there is a subsequence (x_{n_k}) having a limit x' in X_0. Since F is continuous, $y'=F(x')$. Hence the set Y_0 is closed. It is also open, since the transformation is invertible near each of its points. Hence $Y_0=\mathfrak{Y}$. If the inverse transformation were not single-valued, there would be a curve B in X_0 joining two distinct points whose image C in the space $\mathfrak{Y}$ is a closed curve. By a continuous deformation of the curve C into a point the curve B must also be deformed into a point, which easily leads to a contradiction. Essentially this theorem and proof were given by Paul Levy in 1920 [**23**]. Caccioppoli has recently given an exposition of it in general form [**5**].

Schauder has recently proved theorems showing that under certain circumstances the uniqueness of the solution of an equation $y=G(x)$ implies the existence of a solution (x, y) for every y near y_0, where (x_0, y_0) is a solution. One of his theorems is as follows [**27**, p. 686]. Let there be defined in the space $\mathfrak{X}$ a weak convergence satisfying the postulates set down in Part I. Suppose also that every bounded set is weakly compact. Let F and H be transformations defined on an open set X_0 of $\mathfrak{X}$, and let F be totally continuous, in the sense that it transforms a weakly convergent sequence into a sequence converging according to the norm. Suppose that H satisfies a Lipschitz condition with constant $k<1$, and transforms a weakly convergent sequence into a weakly convergent sequence. Then if $G(x)\equiv x+F(x)+H(x)$ establishes a one-to-one correspondence between the open set X_0 and the transformed set $G(X_0)$, $G(X_0)$ is also an open set. Again the proof is made by constructing approximating transformations defined in subspaces of a finite number of dimensions. Schauder applied this theorem to show that in case a partial differential equation of elliptic type

$$F(x, y, z, p, q, r, s, t) = \psi(x, y), \qquad (z = \phi \text{ on the boundary}),$$

is known to have at most one solution, and if it has a solution for $\psi=\psi_0$, $\phi=\phi_0$, then it has a solution for ψ, ϕ near ψ_0, ϕ_0.

Another implicit function theorem has been given by Leray and Schauder [22] for equations of the form

$$x = F(x, k). \tag{13}$$

Let $\overline{R}$ be the closure of a bounded open set R in a space $\mathfrak{X}$, and let K be a finite closed segment of the real axis. Let the transformation F be defined and continuous on $\overline{R}K$, and continuous in k uniformly with respect to x. Moreover, let the set of functional values $F(\overline{R}, K)$ be compact, and suppose equation (13) has no solution with x on the boundary of R. Suppose that at an initial point k_0 of the interval K, the equation (13) has a finite number of solutions $x=x_j$, and that the sum of the "indices" of these solutions is not zero. This last hypothesis is satisfied, for example, in case there are an odd number of solutions near each of which the equation $y=x-F(x, k_0)$ has at most one solution. Then equation (13) has a continuum of solutions (x, k) on which k takes all the values of the interval K. This theorem has the advantage of being a theorem in the large, and of not requiring uniqueness of the initial solution, but the other hypotheses are rather stringent. The proof again depends on transformations F_ϵ which approximate the transformation F on subsets of R of a finite number of dimensions.

All of the theorems which have been quoted have their applications to boundary value problems for differential equations and to integral equations, both linear and non-linear. The field seems to be a fruitful one for further study.

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ON A THEOREM OF PLESSNER

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Plessner‡ has shown that if $f(x) \subset L_2$ on $(-\pi, \pi)$ and

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

then

$$\sum_{n=2}^{\infty} (a_n \cos nx + b_n \sin nx)(\log n)^{-1/2} \tag{1}$$

converges almost everywhere on $(-\pi, \pi)$. We designate the set where (1) converges by $E(Pl, f)$. This set is then known to be of measure 2π. The sets $E(F, f)$, consisting of the points where

$$\phi(t) = f(x + t) + f(x - t) - 2f(x) \to 0, \text{ as } t \to 0,$$

and $E(L, f)$, consisting of the points where

$$\Phi(t) = \int_0^t |\phi(\tau)| \, d\tau = o(t), \text{ as } t \to 0,$$

are of much importance in the theory of Fourier series. The set $E(L, f)$ is known to be of measure 2π for all integrable functions. It is obvious that

$$E(F, t) \subset E(L, f).$$

We propose in this note to investigate the inclusion relationships between these sets and $E(Pl, f)$. We shall prove

$$E(F, f) \not\subset E(Pl, f), \tag{2}$$

and

$$E(Pl, f) \not\subset E(L, f). \tag{3}$$

We first consider (2). Plessner§ showed that, if (1) converges,

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‡ A. Plessner, Journal für Mathematik, vol. 155 (1926), pp. 15–25.

§ Loc. cit., p. 22.

$$(4)\quad S_n(x) = \frac{a_0}{2} + \sum_{\nu=1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x) = o\{(\log n)^{1/2}\}.$$

However, it is well known† that, for a continuous function, the estimate

$$(5)\quad S_n(x) = o(\log n)$$

cannot be improved. This implies that (4) need not be satisfied at every point of continuity and hence

$$E(F, f) \not\subset E(Pl, f).$$

In order to prove (3) we shall construct a function $f(x) \subset L_2$ on $(-\pi, \pi)$ for which (1) converges at $x = 0$ but such that

$$(6)\quad \int_0^t |\phi(\tau)|\, d\tau \neq o(t) \text{ as } t \to 0.$$

The function is similar to one constructed by Paley‡ for another purpose. We define $f(x)$ by

$$f(x) = \begin{cases} x\{(x - n^{-1})n \log n\}^{-1}, \text{ if } n^{-2} \geqq |x - n^{-1}| \geqq n^{-3}, \\ \qquad\qquad\qquad\qquad\qquad\qquad (n = 3, 4, \cdots), \\ 0, \text{ elsewhere on } (0, \pi), \\ f(-x) \text{ for } 0 \geqq x \geqq -\pi. \end{cases}$$

Then, since

$$\int_0^\pi |f(x)|^2 dx = O\left\{\sum_{n=3}^{\infty} n^{-4}(\log n)^{-2} \int_{n^{-2}}^{n^{-2}} \frac{dx}{x^2}\right\}$$

$$= O\left\{\sum_{n=1}^{\infty} n^{-1}(\log n)^{-2}\right\},$$

$f(x) \subset L_2$ on $(-\pi, \pi)$. We have at $x = 0$, $\phi(t) = 2f(t)$ and

$$\int_0^t |\phi(t)|\, dt > \sum_{n=[1/t]+1}^{\infty} (n^2 \log n)^{-1} \int_{n^{-2}}^{n^{-2}} \frac{dx}{x} = \sum_{n=[1/t]+1}^{\infty} n^{-2} > \frac{t}{3},$$

† P. Du Bois Reymond, Abhandlungen der Bayerische Akademie, vol. 12, part 2 (1876). There is a simplification of Du Bois Reymond's method given by Lebesgue, *Leçons sur les Séries Trigonométriques*, 1906, pp. 84–86.

‡ R. E. A. C. Paley, Proceedings Cambridge Philosophical Society, vol. 26 (1930), pp. 173–203; see §10, pp. 201–203.

for t sufficiently small. Now we consider $S_m(0)$. It is well known that

$$S_m(0) = \frac{1}{2\pi}\int_0^\pi \phi(t)\,\frac{\sin\,(m+1/2)t}{\sin\,t/2}\,dt$$
$$= \frac{1}{\pi}\int_0^\pi \phi(t)\,\frac{\sin\,mt}{t}\,dt + O(1),$$

while

$$S_m^*(0) \equiv \frac{1}{\pi}\int_0^\pi \phi(t)\,\frac{\sin\,mt}{t}\,dt$$
$$= \frac{1}{\pi}\sum_{n=3}^{\infty}\frac{2}{n\log n}\left\{\int_{n^{-1}-n^{-3}}^{n^{-1}-n^{-2}}\frac{\sin\,mt}{(t-n^{-1})}\,dt\right.$$
$$\left.+\int_{n^{-1}+n^{-3}}^{n^{-1}+n^{-2}}\frac{\sin\,mt}{(t-n^{-1})}\,dt\right\}.$$

But

$$\sin\,mt = \sin\,(mn^{-1})\cos\,(m(t-n^{-1}))$$
$$+\sin\,(m(t-n^{-1}))\cos\,(mn^{-1}),$$

so that

$$\pi S_m^*(0) = 4\sum_{n=3}^{\infty}(n\log n)^{-1}\cos\,(mn^{-1})\int_{n^{-3}}^{n^{-2}}\frac{\sin\,mt}{t}\,dt$$
$$= 4\left\{\sum_{n=3}^{[m^{1/3}]}+\sum_{n=[m^{1/3}]+1}^{[m^{1/2}]}+\sum_{n=[m^{1/2}]+1}^{\infty}\right\}$$
$$\equiv I_1+I_2+I_3.$$

Now, if $ma<1$, $a>b>0$,

$$\int_b^a\frac{\sin\,mt}{t}\,dt = \int_{mb}^{ma}\frac{\sin\,mt}{t}\,dt = O\{ma\},$$

and, if $mb>1$, $a>b>0$,

$$\int_b^a\frac{\sin\,mt}{t}\,dt = \int_{mb}^{ma}\frac{\sin\,t}{t}\,dt = O\left\{\frac{1}{mb}\right\} = O(1).$$

Hence

$$I_1 = O\left\{\sum_{n=3}^{[m^{1/3}]}(n\log n)^{-1}\frac{n^2}{m}\right\} = O\left\{\frac{1}{m}\sum_{n=3}^{[m^{1/3}]}\frac{n^2}{\log n}\right\} = o(1),$$

$$I_2 = O\left\{\sum_{n=[m^{1/2}]+1}^{[m^{1/2}]}(n\ \log\ n)^{-1}\right\}$$

$$= O\{\log\log m^{1/2} - \log\log m^{1/3}\} = O(1)$$

$$I_3 = O\left\{m\sum_{n=[m^{1/2}]+1}^{\infty}(n\log n)^{-1}n^{-2}\right\}$$

$$= o\left\{m\sum_{n=[m^{1/2}]+1}^{\infty}n^{-3}\right\} = o(1).$$

Therefore

$$S_m(0) = S_m^*(0) + O(1) = O(1).$$

We now apply Abel's partial summation to (1) and get

$$\sum_{n=2}^{\infty}(S_n - S_{n-1})(\log n)^{-1/2}$$

$$= S_1(\log 2)^{-1/2} + \sum_{n=2}^{\infty}S_n\{(\log n)^{-1/2} - [\log(n+1)]^{-1/2}\}, \tag{7}$$

since $S_m(0) = O(1)$ and $(\log n)^{-1/2}\to 0$ as $n\to\infty$. But since $\log n$ is monotone,

$$\sum_{n=2}^{\infty}\{\,|\,(\log n)^{-1/2} - (\log[n+1]^{-1/2}|\,\} = (\log 2)^{-1/2},$$

and therefore (7) converges. This means that the point $x=0$ is contained in $E(Pl, f)$. But since we have already seen that the point $x=0$ is not contained in $E(L, f)$, this proves that

$$E(Pl, f) \not\subset E(L, f).$$

YALE UNIVERSITY

A REMARK ON METHOD IN TRANSFINITE ALGEBRA†

BY MAX ZORN‡

The theorems of Steinitz concerning algebraic closure and the degree of transcendence are barred, from the algebraic point of view, by the well-ordering theorem and its theory. We wish to show how, by introducing a certain axiom on sets of sets instead of the well-ordering theorem, one is enabled to make the proofs shorter and more algebraic. The proofs will be given in terms of the non-axiomatic standpoint of set theory.

DEFINITION 1. A set $\mathfrak{B} = \{B\}$ of sets B is called a *chain*, if for every two sets B_1, B_2, either $B_1 \supset B_2$ or $B_2 \supset B_1$.

DEFINITION 2. A set $\mathfrak{A}$ of sets A is said to be closed (right-closed), if it contains the union $\sum_{\mathfrak{B} \ni B} B$ of every chain $\mathfrak{B}$ contained in $\mathfrak{A}$.

Then our *maximum principle* is expressible in the following form.

(MP). *In a closed set $\mathfrak{A}$ of sets A there exists at least one, A^*, not contained as a proper subset in any other $A \in \mathfrak{A}$.*

APPLICATIONS. I. Let $\mathfrak{R}$ be a ring with a unity element 1; let $\mathfrak{A}$ be the set of all ideals $\mathfrak{a}$ (i) not containing 1 as an element, (ii) containing a certain ideal $\mathfrak{r} \neq \mathfrak{R}$. The set $\mathfrak{A}$ is obviously closed; the maximum principle implies the existence of a maximal ideal $\mathfrak{p}$ with $\mathfrak{r} \subseteq \mathfrak{p} \subset \mathfrak{R} \neq \mathfrak{p}$; this ideal is a prime ideal and the residue class ring $\mathfrak{R}/\mathfrak{p}$ is a *field*.

II. If k is a real field, that is, a field such that no sum of squares vanishes unless all the squares vanish, and K is an arbitrary algebraical extension, then the set of all real fields between k and K is closed, so the MP assures the existence of a *maximal* real field between k and K. In particular, if K is algebraically closed, we obtain a real closure of k.

III. Let K be an arbitrary field extension of k. A set of K-elements $\{a\}$ is said to be algebraically (respectively, linearly) independent, if no finite subset $a_1, a_2, a_3, \cdots, a_n$ satisfies an

† Presented to the Society, October 27, 1934.

‡ Sterling Fellow.

algebraic (linear) relation with coefficients in k (not all vanishing). The set of all these independent sets is closed; a maximal independent system, which exists as a consequence of the MP, is a basis for K/k with respect to algebraic (linear) dependence. (The theorems concerning dependence are consequences of the dependence axioms† only.)

We shall first prove the existence of an algebraic, algebraically closed field extension. Indeed, we shall construct for each set of polynomials $\{p(x)\}$ with coefficients in k a minimal decomposition field. (Let p have the highest coefficient 1 and degree n_p.)

We form the field k' of all rational functions in x over k. Over k' we consider the domain of all polynomials in the following set of indeterminates: $y_{p,i}$, the first index p ranging over all polynomials of our set, the second over the integers from 1 to n_p.

In this domain of integrity let us take the ideal generated by the polynomials

$$\{p(x) - (x - y_{p,1})(x - y_{p,2}) \cdots (x - y_{p,n_p})\}.$$

Suppose that this ideal does not contain the number 1. Then we have a prime ideal containing it, and the residue classes of the indeterminates y generate a minimal field over k, in which all polynomials $p(x)$ are completely decomposed.

If now this ideal should contain the unity element, we should have a representation of 1 as a linear combination of a finite set of differences $p(x)-(x-y_{p,1}) \cdots$ with coefficients rational in x and integral in the y's. This relation in indeterminates y gives the contradiction 1 equal to 0, if we substitute for the indeterminates the roots of the *finite* set of polynomials p occurring in the relation; these roots exist in accordance with *elementary* theorems in a suitable field extension.

Let K_1, K_2 be two algebraic extensions of k, which are minimal decomposition fields for the set of polynomials $\{p(x)\}$ over k. We consider the partial realizations of the uniqueness theorem, that is, the isomorphisms between fields K_1^2 and K_2^2, which are the identity in the common subfield k, K_i^2 lying in K_i. The function tables of these isomorphisms, considered as sets of pairs, form a *closed* system (of sets of pairs); therefore a maximal one must exist in accordance with MP.

† Van der Waerden, *Moderne Algebra*, vol. 1, p. 204.

This maximal isomorphism must range over the entire fields K_i. For let K_i^0 be the fields connected by the maximal isomorphism. Take an arbitrary polynomial $p(x)$ in the set $\{p\}$. The minimal decomposition fields for this polynomial $p(x)$ over K_i^0, generated by adjoining respectively the roots in K_i, are *isomorphic* under an isomorphism which continues our maximal isomorphism between K_1^0 and K_2^0. (As in the case of the existence these proofs are based on the corresponding *finite* theorems; precisely as in the proof dealing with the degree of transcendence.) Hence the decomposition fields reduce to K_i^0; K_i^0 decomposes all polynomials of the set and is therefore identical with K_i; our maximal isomorphism yields the uniqueness of the minimal decomposition field, in particular, the uniqueness of algebraic closure.

Finally we prove the existence of the degree of transcendence. Let $\{x\}$ and $\{y\}$ be two bases of K/k. Once more we consider the partial realizations of our theorem. Namely, we take a (1, 1) correspondence between subsets $\{x\}'$ and $\{y\}'$ of $\{x\}$ and $\{y\}$, where $\{x\}'$ and $\{y\}'$ are *dependent.*

The function tables of these correspondences, considered as sets of pairs, form a *closed* system. We shall prove by the corresponding *finite* theorem concerning degree of transcendence and the axiom of *choice* that a maximal correspondence of this kind which exists in accordance to MP must range over the entire bases $\{x\}$ and $\{y\}$. Take a correspondence with $\{x\}'$ different from $\{x\}$. Take a finite set of x's not contained in $\{x\}'$. This will depend on $\{y\}'$ and a certain finite set of y's which is not contained in $\{y\}'$. This finite set of y's depends on $\{x\}'$, the arbitrarily chosen finite set of x's and a certain other finite set of new x's; proceeding in zigzag we obtain (choice axiom) two sequences of new x's and y's; we may proceed similarly, in case of a finite number (finite theorem) or an infinite (denumerable) set. It is not necessary to show the utility of the MP in further algebraic or analytic examples.

In another paper I shall discuss the relations between MP, the axiom of choice, and the well-ordering theorem. I shall show that they are equivalent if the axiom yielding the set of all subsets of a set is available. It seems that without this axiom the MP is weaker than the well-ordering theorem; the situation may be studied by means of other axioms.

In the present paper all the axioms concerning sets are used;

in the case of algebraic closure there is a stronger result of Hollkott (Hamburg): the axiom of choice is sufficient for the existence and uniqueness of algebraic closure.

An essential simplification is made possible for Baer's† theory of the degree of algebraic extensions. I plan to show elsewhere how the generalized continuum hypothesis may be avoided.

YALE UNIVERSITY

CONCERNING TWO INTERNAL PROPERTIES OF PLANE CONTINUA‡

BY R. E. BASYE

Theorem 1 below was suggested to me by R. L. Moore. Theorem 2 is an extension of Kuratowski's result§ that if three compact plane continua have a point in common and their sum separates a point A from a point B in the plane, then there exists a pair of these continua whose sum separates A from B in the plane. Another extension of this result along combinatorial lines has been given‖ by Čech.

THEOREM 1. *Let H and K be two mutually exclusive and closed subsets of a compact continuum M which lies in the plane. If for each pair of points A and B in H and K, respectively, there exists a finite collection Γ_{AB} of continua in M such that Γ^*_{AB} separates A from B in M, then there exists a finite collection Γ of continua in M such that Γ^* separates H from K in M.*

Let $\epsilon_1, \epsilon_2, \cdots$, be a sequence of positive numbers converging monotonically to zero, with ϵ_1 less than half the distance from H to K. For each i let D_H^i be a domain containing H such that (1) the boundary β_H^i of D_H^i is the sum of a finite number of mutually exclusive simple closed curves, and (2) each point of

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† *Eine Anwendung der Kontinuumhypothese in der Algebra.* Journal für Mathematik, vol. 162.

‡ Presented to the Society, April 6, 1935, under a somewhat different title.

§ Kuratowski, *Théorème sur trois continus,* Monatshefte für Mathematik und Physik, vol. 36 (1929), pp. 77–80.

‖ E. Čech, *Trois théorèmes sur l'homologie,* Publications de la Faculté des Sciences de L'Université Masaryk, No. 144, 1931, pp. 1–21.

D_H^i lies at a distance less than ϵ_i from H. For each i let D_K^i be a similar domain containing K and having boundary β_K^i. We shall show that there exists a value of i such that the number of components of $M-M\cdot(D_H^i+D_K^i)$ which intersect both β_H^i and β_K^i is finite.

Suppose there are infinitely many such components for each value of i. Consider for each i an infinite sequence $C_1^i, C_2^i, \cdots$, of components of $M-M\cdot(\overline{D_H^i}+\overline{D_K^i})$, no two of which lie in the same component of $M-M\cdot(D_H^i+D_K^i)$, and such that both β_H^i and β_K^i contain a limit point of C_j^i $(j=1, 2, \cdots)$. There exists an infinite subsequence $\alpha^i=E_1^i, E_2^i, \cdots$ of the sequence $\overline{C_1^i}, \overline{C_2^i}, \cdots$, and two components j_H^i, j_K^i of β_H^i, β_K^i, respectively, such that each element of α^i intersects both j_H^i and j_K^i.

No component of β_H^i or β_K^i distinct from j_H^i and j_K^i can intersect more than two elements of α^i. For suppose j is such a component, intersecting, say, the three elements E_1^i, E_2^i, E_3^i of α^i. The continua E_1^i, E_2^i, E_3^i are mutually exclusive, intersect j, and are not disconnected by j. It follows that no two points of $j_H^i\cdot E_n^i$, $(n=1, 2, 3)$, are separated from each other in j_H^i by $\sum_{r=1}^3 E_r^i-E_n^i$. Hence there exist three arcs b_1, b_2, b_3 such that (1) $b_1+b_2+b_3=j_H^i$, and (2) b_n, $(n=1, 2, 3)$, contains $j_H^i\cdot E_n^i$ but no point of $\sum_{r=1}^3 E_r^i-E_n^i$. Denote by F_n^i, $(n=1, 2, 3)$, the continuum $E_n^i+b_n+j$. Let δ_H^i and δ_K^i denote the complementary domains of j_H^i and j_K^i which do not contain j_K^i and j_H^i, respectively. The three continua F_n^i, $(n=1, 2, 3)$, have a point in common and their sum separates a point X of δ_H^i from a point Y of δ_K^i in the plane. Hence, by a theorem of Kuratowski,† there exist two of these continua, say F_1^i and F_2^i, such that $F_1^i+F_2^i$ separates X from Y in the plane. But $\delta_H^i+(E_3^i-j\cdot E_3^i)+\delta_K^i$ is a connected set which contains X and Y but no point of $F_1^i+F_2^i$.

Consequently, the number of components of $\beta_H^i+\beta_K^i$ being finite, there exists an integer r such that each element of the sequence $\gamma^i=E_r^i, E_{r+1}^i, \cdots$, intersects both components j_H^i, j_K^i without meeting any other component of $\beta_H^i+\beta_K^i$. The intersection of each element E_n^i of γ^i with j_H^i is contained in an arc (or point) of j_H^i which intersects no other element of γ^i. Denote by a_n^i the minimal such arc (or point), and by b_n^i a

† Loc. cit.

similar arc (or point) containing the intersection of E_n^i with j_K^i. The sequence $\{a_n^i+b_n^i\}$, $(n=r, r+1, \cdots)$, contains a subsequence which has a sequential limiting set; let P_i and Q_i denote points of this limiting set which lie on j_H^i and j_K^i respectively. The sequence $\{P_i+Q_i\}$ contains a subsequence which has a sequential limiting set; let P and Q denote points of this limiting set which lie in H and K, respectively.

By hypothesis there exists a finite collection Γ_{PQ} of continua in M such that Γ_{PQ}^* separates P from Q in M. Hence $M-\Gamma_{PQ}^* = N_P+N_Q$, where N_P and N_Q are mutually separated sets containing P and Q, respectively. Let R_P and R_Q be regions containing P and Q, respectively, such that $R_P\cdot(N_Q+\Gamma_{PQ}^*)=R_Q\cdot(N_P+\Gamma_{PQ}^*)=0$. Let i be chosen so that P_i and Q_i lie in R_P and R_Q, respectively. Then there exists an infinite subsequence $\epsilon^i=E_{n,1}^i$, $E_{n,2}^i, \cdots$ of γ^i such that $a_{n,t}^i$ and $b_{n,t}^i$, $(t=1, 2, \cdots)$, are subsets of R_P and R_Q, respectively. Every element of ϵ^i intersects some element of Γ_{PQ}. Therefore some element W of Γ_{PQ} intersects at least two elements ϵ^i, hence contains a point of $\beta_H^i+\beta_K^i$. Hence, if $E_{n,s}^i$ denotes an element of ϵ^i which intersects W, then W contains a point of $a_{n,s}^i+b_{n,s}^i$. But $a_{n,s}^i+b_{n,s}^i$ is a subset of R_P+R_Q, which contains no point of Γ_{PQ}^*.

It follows that there exists a value of i such that the collection Γ of components of $M-M\cdot(D_H^i+D_K^i)$ which intersect both β_H^i and β_K^i is finite. Denote by U_H the sum of those components of $M-M\cdot(D_H^i+D_K^i)$ which intersect β_H^i but not β_K^i, and by U_K the sum of those components which intersect β_K^i but not β_H^i. It can be seen that $M\cdot D_H^i+U_H$ and $M\cdot D_K^i+U_K$ are mutually separated sets containing H and K, respectively. Hence Γ^* separates H from K in M.

R. L. Moore has proved† the closely related result whose statement is identical with that of Theorem 1 except that each Γ_{AB} and Γ are taken as finite sets. The argument is valid in any space satisfying his axioms 0 and 1. An example showing that Theorem 1 is not true for three-dimensional continua has been given by W. T. Reid.‡

The conclusion of Theorem 1 will also hold if Γ_{AB}^* merely

† *Foundations of Point Set Theory*, Colloquium Publications of this Society, vol. 13, p. 140, Theorem 57.

‡ See this issue of this Bulletin, pp. 683–684.

weakly disconnects† A from B in M, provided M is locally connected at each point of $H+K$. The argument is for the most part similar.

Again, if we require only that Γ_{AB} be a countable collection of continua in M such that Γ_{AB}^* is closed and separates A from B in M, then there exists a countable collection Γ of continua in M such that Γ^* is closed and separates H from K in M.

THEOREM 2. *Let M be any subcontinuum of a plane or a sphere and let Z be any subset (possibly vacuous) of M. If G is a finite collection of connected subsets of M such that* (1) *there exist r points (r finite) whose sum L intersects each element of G, and* (2) *G^*+Z weakly disconnects a point A from a point B in M, then there exists a subcollection H of G, containing not more than $2r$ elements, such that H^*+Z weakly disconnects A from B in M.*

The case where M is a subset of a plane is a consequence of the case where M is a subset of a sphere. Suppose, then, that M is a subcontinuum of a sphere S. We shall discuss in order the three cases that may arise.

CASE 1. Suppose $r=1$. Assume the theorem false. Let $g_1, \cdots, g_n$ denote the elements of G and let O be a point common to these elements. There exist continua C_{ij}, $(i, j=1, \cdots, n)$, such that C_{ij} contains $A+B$ and has no point in common with g_i+g_j+Z. Let D_k, $(k=1, \cdots, n)$, be a connected domain which contains g_k and contains no point of any C_{kj}, $(j=1, \cdots, n)$. Let D_0 be a domain which contains Z but contains no point of any C_{ij}, $(i, j=1, \cdots, n)$. Let $\Delta=\sum_{i=0}^{n} D_i$. The domain $(S-M)+\Delta$ weakly disconnects, hence separates, A from B in S. Hence there exists a component Q of $(S-M)+\Delta$ which separates A from B in S, and Q must contain the connected set $\sum_{i=1}^{n} D_i$.

If Q_i, $(i=1, \cdots, n)$, denotes that component of $(S-M)+D_0+D_i$ which contains g_i, then $Q=Q_1+\cdots+Q_n$. For suppose the contrary and let R denote the set $Q-(Q_1+\cdots+Q_n)$. Since Q_1+R is not connected, R contains a subset R_1 such that R_1 and $(Q_1+R)-R_1$ are mutually separated sets. In general, if R_i has been defined, R_{i+1}, $(i=1, \cdots, n-1)$, will

† A set L weakly disconnects a set A from a set B in a connected set M if it contains a point of every connected and relatively closed subset of M which intersects both A and B.

denote a subset of R_i such that R_{i+1} and $(Q_{i+1}+R_i)-R_{i+1}$ are mutually separated sets. It follows that R_n and $Q-R_n$ are mutually separated sets, contrary to the connectedness of Q.

Let F_{it}, $(i=1, \cdots, n;\ t=1, 2, \cdots)$, be a subcontinuum of Q_i which contains O and every point of Q_i whose distance from the boundary of Q_i is greater than $1/t$. Let $F_t=F_{1t}+\cdots+F_{nt}$, $(t=1, 2, \cdots)$. There exists a value w of t such that F_w separates A from B in S. For on the contrary supposition there exists a sequence of continua $N_1, N_2, \cdots$, such that N_t, $(t=1, 2, \cdots)$, contains $A+B$ and contains no point of F_t. The limiting set of this sequence is a continuum which contains $A+B$ but no point of $Q_1+\cdots+Q_n$. But this is impossible. Consequently† there exist two of the continua F_{iw}, say F_{1w} and F_{2w}, whose sum separates A from B in S. Hence Q_1+Q_2 separates A from B in S. But the continuum C_{12} contains $A+B$ and has no point in common with Q_1+Q_2.

Case 2. Suppose that $r>1$ and G^* is connected. There exists a subcollection K of G having not more than $2r-2$ elements and such that K^* is a connected set which contains L. Denote by G' the collection of all elements each of which is the sum of K^* and an element of G. The elements of G' are connected sets having K^* in common; hence, by Case 1, there exist two elements g_1' and g_2' of G' such that $g_1'+g_2'+Z$ weakly disconnects A from B in M. The set $g_1'+g_2'$ is the sum of $2r$ or less elements of G.

Case 3. Suppose merely that $r>1$. Let H be a subcollection of G which is irreducible with respect to the property that H^*+Z weakly disconnects A from B in M. Let $C_1, \cdots, C_q$ denote the components of H^* and let r_i, $(i=1, \cdots, q)$, be the number of points common to L and C_i. By Case 2 there exist $2r_i$ or less elements of H lying in C_i, $(i=1, \cdots, q)$, such that, if T_i denotes their sum, $T_i+[Z+(H^*-C_i)]$ weakly disconnects A from B in M. Hence C_i cannot contain more than $2r_i$ elements of H. Therefore, since $r \geqq r_1+\cdots+r_q$, H^* cannot contain more than $2r$ elements of G.

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† C. Kuratowski, loc. cit.

A NEW UNIVERSAL WARING THEOREM FOR EIGHTH POWERS

BY ALVIN SUGAR

1. *Introduction.* Hardy and Littlewood* in their proof of Waring's theorem obtained a constant $C=C(s, k)$ beyond which every number is a sum of s integral kth powers $\geqq 0$. Recently Dickson perfected an algebraic method by which he was able to show that every positive integer $\leqq C$ is a sum of s integral kth powers $\geqq 0$. Thus we are now able to obtain universal Waring theorems for relatively small values of s.

We shall consider in this paper the problem of meeting the Hardy and Littlewood constant by Dickson's method and establishing a new universal Waring theorem for eighth powers. The earlier result for eighth powers was 575, obtained by Dickson.†

2. *Proof of the Principal Theorem.* We write

$$(1) \qquad a = 2^8, \qquad b = 3^8, \qquad c = 4^8, \qquad d = 5^8, \qquad e = 6^8.$$

The right side of

$$m = n + Aa + Bb + \cdots + Qq, \quad (n, A, B, \cdots, Q \text{ integral}),$$

is a *resolution* of m of *weight* $w(m) = n+A+B+\cdots+Q$. When $n, A, B, \cdots, Q \geqq 0$ the resolution is a *decomposition.*

By division we obtain

$$(2) \quad b = 161 + 25a, \; c = -74 + 10b, \; d = 56 + 15a + 9b + 5c,$$

$$(3) \qquad e = 21 + 22a + 7b + c + 4d.$$

Consider an integer M, such that $2d+e \leqq M \leqq 3d+e$. We can express the integer $P = M-2d-e$ uniquely in the form $R+N$, where

$$(4) \qquad 0 \leqq R < a = 256, \qquad N = Aa + Bb + Cc,$$

$$(5) \quad C = [P/c],‡ \; B = [(P - Cc)/b], \; A = [(P - Bb - Cc)/a].$$

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* A simplified proof can be found in Landau, *Vorlesungen über Zahlentheorie*, vol. 1, 1927, pp. 235–360.

† This Bulletin, vol. 39 (1933), p. 713.

‡ $[x]$ denotes the largest integer $\leqq x$.

Since $N<d$, we obtain by (4) and (5) the inequalities $Cc<d$, $Bb<c$, and $Aa<b$. Hence

$$0 \leqq A < 26, \qquad 0 \leqq B < 10, \qquad 0 \leqq C < 6. \tag{6}$$

Since

$$M = R + Aa + Bb + Cc + 2d + e, \tag{7}$$

then

$$\begin{aligned} w(M) &= R + A + B + C + 3 \\ &\leqq 255 + 25 + 9 + 5 + 3 = 297. \end{aligned} \tag{8}$$

Since (7) defines a decomposition of M, we can state the following lemma.

LEMMA 1. *Every integer M, such that $2d+e \leqq M \leqq 3d+e$, is a sum of 297 eighth powers.*

Let us now consider the problem of obtaining a smaller value for $w(M)$. Table I contains a list of certain equations of the form

$$r = A'a + B'b + C'c + D'd + E'e. \tag{9}$$

Such an equation defines a resolution of r of weight w. We shall refer to an equation of Table I by citing its r value whenever we may do so without ambiguity. For example, equation 31, $31 = -10a+6c-d$, which defines a resolution of 31 of weight -5, is the first equation listed in Table I. We can readily verify these equations by (1).

We write (7) in the form $M_1 = A_1a+B_1b+C_1c+2d+e+r+r'$, A_1, B_1, C_1 fixed and $R=r+r'$, where $0 \leqq r' < r_f - r$ (the subscript f is used to denote that r_f is the equation immediately following r in Table I and possessing the property $r_f > r$). Eliminating r between this equation and (9), we obtain

$$\begin{aligned} M_1 = {} & (A_1 + A')a + (B_1 + B')b + (C_1 + C')c \\ & + (2 + D')d + (1 + E')e + r'. \end{aligned} \tag{10}$$

We construct a tablette $A=A_1$, $B=B_1$, $C=C_1$ by listing the r and w values of resolutions of Table I whose coefficients satisfy the inequalities $A_1+A' \geqq 0$, $B_1+B' \geqq 0$, $C_1+C' \geqq 0$ (it should be noted that for such resolutions, (10) gives decompositions of M_1 since all the resolutions in Table I satisfy the inequalities

TABLE I*

LIST OF EQUATIONS

r	w	a	b	c	d	e	r	w	a	b	c	d	e
31	−5	−10	0	6	−1	0	140	9	−3	8	1	4	−1
41	23	2	7	13	2	−1	148	18	0	20	−2	0	0
62	−10	−20	0	12	−2	0	157	15	6	−1	12	−2	0
66	0	−3	−2	2	4	−1	158	46	12	27	5	3	−1
70	48	31	8	11	−2	0	169	34	26	9	−1	0	0
72	18	−8	7	19	1	−1	171	4	−13	8	7	3	−1
81	58	1	47	9	2	−1	189	41	2	27	11	2	−1
84	37	12	17	6	3	−1	192	20	13	−3	8	3	−1
95	25	26	−1	0	0	0	200	29	16	9	5	−1	0
97	−5	−13	−2	8	3	−1	210	8	−20	20	10	−2	0
105	4	−10	10	5	−1	0	214	18	−3	18	0	4	−1
105	53	38	6	7	3	−1	223	15	3	−3	14	2	−1
115	32	2	17	12	2	−1	231	24	6	9	11	−2	0
126	20	16	−1	6	−1	0	245	13	−13	18	6	3	−1

$2+D' \geqq 0$, $1+E' \geqq 0$). From the r and w columns we form a new column, the W column, where $W=(r_f-1)-r+w$. We obtain our tablette in its final form by deleting all equations r_f for which $w_f > W$. We denote the greatest W value in this tablette by $G(A_1, B_1, C_1)$. For example, upon constructing the tablette $A=0$, $B=0$, $C=0$ we get

$A=0$,	$B=0$,	$C=0$	$A=0$	$B=0$	$C=0$
r	w	W	r	w	W
0	0	40	115	32	74
41	23	51	158	46	76
70	48	58	189	41	51
81	58	60	200	29	59
84	37	57	231	24	48
105	53	62		$G=76$	

By constructing eleven such tablettes, we obtain

$$
\begin{aligned}
&G(0,0,0)=76, \quad G(3,0,0)=62, \quad G(15,4,0)=41,\\
&G(0,1,0)=60, \quad G(5,4,0)=47, \quad G(20,0,0)=39,\\
&G(0,0,1)=74, \quad G(14,0,0)=50, \quad G(10,0,0)=57.\\
&G(0,0,2)=64, \quad G(20,4,0)=37,
\end{aligned} \tag{11}
$$

* The method of obtaining these equations is analogous to that explained in Dickson's paper on ninth powers in this Bulletin, vol. 40 (1934), pp. 487–493.

Because of the exclusive condition $w_f > W$, the tablettes have the property (P), $G(A_2, B_2, C_2) \leqq G(A_1, B_1, C_1)$ if $A_1 \leqq A_2$, $B_1 \leqq B_2$, $C_1 \leqq C_2$.

Let us now consider $M_1 = R + A_1 a + B_1 b + C_1 c + 2d + e$. Since $w(R) \leqq G(A_1, B_1, C_1)$,

$$w(M_1) \leqq A_1 + B_1 + C_1 + 3 + G(A_1, B_1, C_1).$$

Dropping the subscripts we seek the maximum value H of the function $A+B+C+3+G(A, B, C)$ in the range (6). From (11) and (P) it is evident that H 81. This result may be expressed as follows.

LEMMA 2. *Every integer M, such that* $2{,}460{,}866 \leqq M \leqq 2{,}851{,}491$, *is a sum of* 81 *eighth powers.*

From this interval we ascend to the Hardy and Littlewood constant by employing two theorems of Dickson, Theorems 10 and 12 in this Bulletin.* Using Theorem 10 and a table of eighth powers it was found that 102 eighth powers suffice for the enlarged interval from $q = 2d + e$ to $L_0 = 2{,}235{,}617 \cdot 10^9$. Applying Theorem 12, we obtain

$$\log L_t = (8/7)^t (\log L_0 + h) - h, \qquad h = -8 \log 8,$$

$h = \bar{8}.775280$, $\log(\log L_0 + h) = 0.909806$, $\log(8/7) = 0.057992$.

We take $t = 464$; then $\log \log L_t = 27.818$. Hence 566 eighth powers suffice from q to L_t.

James showed that $\log_e C = 20 \cdot 8^3 2^{\eta}$, where $C = C(s)$ is the Hardy and Littlewood constant, s is the number of eighth powers, and

$$\eta = \frac{20.1s - 162}{s - 426}.$$

For $s = 566$ we obtain $\log \log C = 27.762$. Hence every integer $> q$ is a sum of 566 eighth powers. It is evident from (8) that 294 eighth powers suffice from 0 to d. Consequently 300 eighth powers suffice from 0 to $7d$. Since $q < 7d$, we have the following theorem.

THEOREM. *Every positive integer is a sum of* 566 *integral eighth powers.*

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* Loc. cit., vol. 39 (1933), pp. 710–711.

SOME THEOREMS ON TENSOR DIFFERENTIAL INVARIANTS

BY JACK LEVINE

1. *Introduction.* In the theory of algebraic invariants there is a theorem which states that if an absolute invariant be written as the quotient of two relatively prime polynomials, then the numerator and denominator are relative invariants.* If we consider absolute scalar differential invariants of a metric (or affine) space, then it is possible to prove a similar theorem regarding them. In the course of the proof we give a new proof of the fact that in a relation of the form (2) the ϕ must be a power of the Jacobian of the coordinate transformation. (In the algebraic theory the u_j^i are of course constants.) This proof involves the use of the differential equations satisfied by the scalar.† In this proof it is not necessary to restrict B and ϕ to be polynomials in their arguments as is done in the usual proof of the corresponding theorem in the invariant theory. It is sufficient to assume that ϕ possesses first derivatives with respect to the u_j^i and that $B(\bar{g})$ is an analytic function of ϵ in the neighborhood of $\epsilon=0$. We also extend the theorem to the case of tensor differential invariants of the form (5).

2. *Scalar Differential Invariants.* We consider the differential invariants of a metric space V_n with a quadratic form $g_{ij}dx^i dx^j$. Let

$$A\left(g_{ij};\ \frac{\partial g_{ij}}{\partial x^k};\ \cdots;\ \frac{\partial^p g_{ij}}{\partial x^k \cdots \partial x^l}\right)$$

be an absolute scalar invariant of V_n which we take to be rational in its arguments. We can then write A in terms of the g_{ij} and their extensions $g_{ij,k\cdots l}$, and we have

$$A(g_{ij};\ 0;\ g_{ij,kl};\ \cdots) = \frac{B(g_{ij};\ 0;\ g_{ij,kl};\ \cdots)}{C(g_{ij};\ 0;\ g_{ij,kl};\ \cdots)},$$

* See, for example, H. W. Turnbull, *The Theory of Determinants, Matrices, and Invariants*, p. 277.

† T. Y. Thomas and A. D. Michal, *Differential invariants of relative quadratic differential forms*, Annals of Mathematics, vol. 28 (1927), p. 679.

where B and C are polynomials in the g_{ij} and their extensions. We may assume that B and C have no common factor. Now

$$\frac{B(g_{ij};\cdots)}{C(g_{ij};\cdots)} = \frac{B(\bar{g}_{ij};\cdots)}{C(\bar{g}_{ij};\cdots)}, \tag{1}$$

under an arbitrary coordinate transformation $x \to \bar{x}$, the barred g's being the g's in the $(\bar{x})$ coordinate system. It is easily shown we must have*

$$B(\bar{g}_{ij};\cdots) = \phi(u_b^a)B(g_{ij};\cdots), \tag{2}$$

$$C(\bar{g}_{ij};\cdots) = \phi(u_b^a)C(g_{ij};\cdots), \tag{3}$$

where $u_b^a = \partial x^a/\partial \bar{x}^b$ and ϕ is a polynomial in the u's. We now prove ϕ is a power of $\left|\partial x/\partial \bar{x}\right|$, thus showing B and C are relative scalars.

Write (2) in the form

$$B(\bar{g})\phi^{-1} = B(g),$$

and consider the infinitesimal transformation

$$x^i = \bar{x}^i + \epsilon\xi^i(\bar{x}).$$

We have

$$\left(\frac{d\phi^{-1}}{d\epsilon}\right)_{\epsilon=0} = -\left(\frac{\partial\phi}{\partial u_j^i}\right)_{\epsilon=0}\left(\frac{du_j^i}{d\epsilon}\right)_{\epsilon=0} = -\left(\frac{\partial\phi}{\partial u_j^i}\right)_{\epsilon=0}\frac{\partial\xi^i(x)}{\partial x^j}.$$

As ϕ is a polynomial, so also is $\partial\phi/\partial u_j^i$, and on evaluating this last expression at $\epsilon=0$, we obtain a set of constants k_i^j, so that

$$\left(\frac{d\phi^{-1}}{d\epsilon}\right)_{\epsilon=0} = -k_i^j\frac{\partial\xi^i}{\partial x^j}.$$

Proceeding then as in the paper by Thomas and Michal, page 679, we obtain the differential equations satisfied by B in the form

$$X_s^t(p)B = k_s^t B. \tag{4}$$

Now for any function G we have†

$$(X_s^t, X_m^l)G = \delta_m^t X_s^l G - \delta_s^l X_m^t G,$$

* H. W. Turnbull, loc. cit.

† Thomas and Michal, loc. cit., p. 663.

and, in particular, for B we would have

$$(X_s^t, X_m^l)B = \delta_m^t X_s^l B - \delta_s^l X_m^t B = B(\delta_m^l k_s^l - \delta_s^l k_m^t).$$

But also

$$(X_s^t, X_m^l)B = X_s^t(k_m^l B) - X_m^l(k_s^t B) = B(k_m^l k_s^t - k_s^t k_m^l) = 0,$$

so that

$$\delta_m^t k_s^l - \delta_s^l k_m^t = 0,$$

from which follows $k_m^l = k\delta_m^l$, where k is a constant. Substituting in (4) we find that B is a relative scalar of weight k and* $\phi = \left|\partial x/\partial \bar{x}\right|^k$. Similar results hold for C also. As stated in §1 we can prove a generalization of this result which we state as follows.

THEOREM 1. *Given a function*

$$B\left(g_{ij}; \frac{\partial g_{ij}}{\partial x^k}; \cdots; \frac{\partial^p g_{ij}}{\partial x^k \cdots \partial x^l}\right),$$

with the law of transformation

$$B\left(\bar{g}_{ij}; \cdots; \frac{\partial^p \bar{g}_{ij}}{\partial \bar{x}^k \cdots \partial \bar{x}^l}\right) = \phi(u_b^a) B\left(g_{ij}; \cdots; \frac{\partial^p g_{ij}}{\partial x^k \cdots \partial x^l}\right),$$

where ϕ possesses first derivatives in the u's and $B(\bar{g})$ is analytic in the neighborhood of $\epsilon = 0$. Then ϕ is a power of the Jacobian and B is a relative scalar differential invariant.

3. *Tensor Differential Invariants.* Consider the absolute tensor differential invariant with components of the form

$$T_{i\cdots j}^{a\cdots b} = \frac{U_{i\cdots j}^{a\cdots b}(g_{kl}; g_{kl,mq}; \cdots)}{D(g_{kl}; \cdots)}, \tag{5}$$

where the U's and D are polynomials (with no common factor) in their arguments. Corresponding to (1) in the scalar case we have

$$\frac{U_{i\cdots j}^{a\cdots b}(g) u_k^i \cdots u_l^j}{U'^{m\cdots p}_{k\cdots l}(g, u) u_m^a \cdots u_p^b} = \frac{D(g)}{D'(g, u)} = \frac{Q(g, u)}{P(g, u)}, \tag{6}$$

* Thomas and Michal, loc. cit.

(P, Q having no common factor), where the primed U's and D represent the result of replacing the barred g's by their values in terms of the g's and u's in the expressions $U^{\cdots}(\bar{g})$ and $D(\bar{g})$. From (6) we see that Q is a factor of D and of $U^{a\cdots b}_{i\cdots j}u^i_k \cdots u^j_l$, and hence must be a function $Q'(g)$, so that

$$U^{a\cdots b}_{i\cdots j}(g)u^i_k \cdots u^j_l = Q'(g)V^{a\cdots b}_{k\cdots l}(g, u), \tag{7}$$

$$D(g) = Q'(g)R(g). \tag{8}$$

In (7) put $u^i_j = \delta^i_j$; then

$$U^{a\cdots b}_{k\cdots l}(g) = Q'(g)V^{a\cdots b}_{k\cdots l}(g, \delta). \tag{9}$$

Hence $Q'(g) = \text{const.}$, since D and the U's have no common factor. Since $D(g)$ and $D'(g, u)$ are of the same degree in the g's, it then follows from (6) that $P(g, u) = P'(u)$, so that

$$D(\bar{g}) = \phi(u^i_j)D(g).$$

Hence as proved in the previous section for B, *we have shown that D is a relative scalar of weight k, and therefore $U^{a\cdots b}_{i\cdots j}$ are the components of a relative tensor of weight k.* We can also prove the following theorem.

THEOREM 2. *If the set of quantities*

$$T^{i\cdots j}_{a\cdots b}\left(g_{kl}; \cdots ; \frac{\partial^p g_{kl}}{\partial x^m \cdots \partial x^r}\right)$$

have the transformation law

$$T^{s\cdots t}_{v\cdots w}(\bar{g}_{kl}; \cdots)u^i_s \cdots u^j_t = \phi(u^d_e)T^{i\cdots j}_{a\cdots b}(g_{kl}; \cdots)u^a_v \cdots u^b_w,$$

then ϕ is a power of $\left|\partial x/\partial \bar{x}\right|$ and the T's are components of a relative tensor invariant, it being assumed that ϕ possesses first derivatives in the u's, and $T(\bar{g})$ are analytic in the neighborhood of $\epsilon = 0$.

The proof is similar to that used for B of the previous section. Similar results to those obtained for metric scalar and tensor differential invariants hold for affine invariants.

PRINCETON UNIVERSITY

A CERTAIN THREE-DIMENSIONAL CONTINUUM*

BY W. T. REID

In this note there is given an example of a bounded continuum in three-dimensional euclidean space such that there exists a point A of M and subcontinua G, K_1, and K_2 of $M-A$ satisfying the following conditions: (1) $M-K_i$, $(i=1, 2)$, is the sum of two mutually separated point sets each of which is connected; (2) each point of G is separated from A in M by either K_1 or K_2; however, (3) there does not exist a point set consisting of a finite number of connected subsets of M and separating A from G in M.

This example was obtained in 1928 while the author was a student at the University of Texas. It answers in the negative for three-dimensional continua a certain question concerning continua proposed to the author by R. L. Moore at that time. R. E. Basye† has recently answered Moore's question in the affirmative for plane continua. It is because of Basye's result that the example of this note is now published.

EXAMPLE. Using rectangular coordinates, consider the points $p_i=(1/i,\ 0,\ 0)$, $p_0=(0,\ 0,\ 0)$, $t_i=(1/i,\ 0,\ 1)$, $t_0=(0,\ 0,\ 1)$, $s_i=(1/i,\ -1,\ 0)$, $s_0=(0,\ -1,\ 0)$, and $A=(0,\ 1,\ 0)$, $(i=1, 2, \cdots)$. If P and Q are two points we shall denote by PQ the closed straight line segment from P to Q. The continuum M is now defined as $M=\sum[Ap_n+p_ns_n+p_nt_n]+s_1s_0+s_0t_0+t_0t_1$, where it is understood that the summation is with respect to n over the values $n=0, 1, 2, \cdots$. The subcontinua G, K_1, and K_2 are defined as follows: $G=s_1s_0+s_0t_0+t_0t_1$, $K_1=\sum p_nt_n+t_0t_1$, $K_2=\sum p_ns_n+s_0s_1$. It is seen that $M-K_1=U_1+V$, where $U_1=\sum[s_np_n-p_n]+[s_0t_0-t_0]+s_0s_1$ and $V=\sum[Ap_n-p_n]$. Similarly, $M-K_2=U_2+V$, where $U_2=\sum[t_np_n-p_n]+[s_0t_0-s_0]+t_0t_1$. Clearly U_i and V, $(i=1, 2)$, are mutually separated point sets each of which is connected. Moreover, each point of G belongs to either U_1 or U_2. However, $M-(G+A)=\sum B_n$, where $B_n=\sum[Ap_n-A]+\sum[p_nt_n-t_n]+\sum[p_ns_n-s_n]$. Since for $n\neq m$, $(n, m=0, 1, \cdots)$,

* Presented to the Society, February 23, 1935.

† See this issue of this Bulletin, pp. 670–674.

the connected point sets B_n and B_m have no point in common, there clearly exists no point set consisting of a finite number of connected subsets of M and separating G from A in M.

THE UNIVERSITY OF CHICAGO

A THEOREM ON PLANE CONTINUA*

BY W. T. REID

1. *Introduction.* In this paper the following theorem is proved.

THEOREM. *If M is a plane continuum, and K is a proper subcontinuum of M, then at least one component of $M-K$ has a limit point in K.*

Two points sets are *mutually separated* if they are mutually exclusive and neither of them contains a limit point of the other. A point set is said to be *connected* if it is not the sum of two non-vacuous mutually separated point sets. A point set which is both connected and closed is a *continuum*. A *component* of a point set N is a connected subset of N which is not a proper subset of any other connected subset of N. The set of all points in the plane will be denoted by S.†

2. *Proof of the Theorem.* If M is a bounded continuum and K is a proper subcontinuum of M, it is well known that every component of $M-K$ has a limit point in K.‡ If M is unbounded then it is no longer true that *every* component of $M-K$ has a limit point in K.§

If K is a bounded subcontinuum of an unbounded plane continuum M, then the above theorem may be proved readily. For

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* Presented to the Society, December 28, 1934. The result of this paper was obtained in 1928, while the author was a student under R. L. Moore at the University of Texas. Recently both R. L. Moore and J. H. Roberts have proved results beyond that of the present paper and have suggested that I publish my original result.

† These definitions are those customarily used in point set theory. See, for example, R. L. Moore, *Foundations of Point Set Theory*, Colloquium Publications of this Society, vol. 13. For brevity, this treatise will be referred to as "Moore."

‡ See, for example, Moore, p. 24.

§ See Moore, p. 25, example 2.

let J be a simple closed curve enclosing K. If D is the interior of J and M_1 denotes the component containing K of the subset of M belonging to $D+J$, then M_1 is a bounded continuum containing K as a proper subcontinuum. As indicated above, if C_1 is a component of M_1-K, then C_1 has a limit point in K. Hence the component C of $M-K$ which contains C_1 has a limit point in K, and the theorem is proved.

Now suppose that K is an unbounded proper subcontinuum of the plane continuum M. If x is a point of $M-K$, the component of $M-K$ containing x will be denoted by C_x. On the assumption that the above theorem is false, it is seen that each component C_x is an unbounded continuum. It follows, therefore, that M is the sum of a set of mutually exclusive unbounded continua consisting of K and the totality of components of $M-K$. This set of mutually exclusive continua is necessarily uncountable in number.*

We shall prove the preceding theorem by showing that the assumption that it is false leads to a contradiction. The proof depends upon the following auxiliary lemma which will be established in the next section.

LEMMA 1. *On the assumption that the above theorem is false, the components C_x of $M-K$ satisfy the following conditions*: (α) *for each component C_x, $M-C_x$ is connected*; (β) *if x_1 and x_2 are two points of $M-K$ such that C_{x_1} and C_{x_2} are mutually exclusive, then there does not exist a simple continuous arc x_1x_2 such that $x_1x_2-(x_1+x_2)$ belongs to $S-M$.*

With the help of this lemma the proof is as follows. Let D be a complementary domain of K containing points of $M-K$. Since M is a continuum, it follows that D must contain infinitely many of the components C_x of $M-K$. Let x_1 and x_2 be points of $M-K$ belonging to D such that C_{x_1} and C_{x_2} are mutually exclusive, and consider a simple continuous arc x_1x_2 which belongs to D.† Let X denote the subset of points x of $M-K$ such that

* See R. L. Moore, *Concerning the sum of a countable number of mutually exclusive continua in the plane*, Fundamenta Mathematicae, vol. 6 (1924), pp. 189–202.

† For a proof that such an arc exists, see R. L. Moore, *On the foundations of plane analysis situs*, Transactions of this Society, vol. 17 (1916), pp. 131–164, in particular, p. 137; see also, Moore, p. 86.

the component C_x has a point in common with the arc x_1x_2. Now if X were connected, it would follow that C_{x_1} and C_{x_2} belong to a single component of $M-K$, which is a contradiction. Hence X is the sum of two mutually separated point sets X_1 and X_2. Clearly, if x is a point of $M-K$ belonging to a set X_i, $(i=1, 2)$, then every point of C_x belongs to the same X_i. Let Y_i, $(i=1, 2)$, denote the set of points common to X_i and the arc x_1x_2. The sets Y_i are seen to be closed and mutually exclusive. There exists, therefore, a subarc $x_{10}x_{20}$ of x_1x_2 such that x_{i0} belongs to Y_i and $x_{10}x_{20}-(x_{10}+x_{20})$ belongs to x_1x_2-X and therefore to $S-M$. This, however, is a contradiction to conclusion (β) of Lemma 1, since x_{10} and x_{20} belong to mutually exclusive components of $M-K$. We have proved that the assumption that our theorem is false leads to a contradiction. Hence the theorem is true.

3. *Proof of Lemma* 1. Conclusion (α) of Lemma 1 has been established by Knaster and Kuratowski.* We shall prove conclusion (β) by showing that the assumption that it is false leads to a contradiction. For suppose that there are two points x_1 and x_2 of $M-K$ such that C_{x_1} and C_{x_2} are mutually exclusive and there exists a simple continuous arc x_1x_2 such that $x_1x_2-(x_1+x_2)$ belongs to $S-M$. Let D denote the complementary domain of the continuum K which contains $C_{x_1}+C_{x_2}+x_1x_2$, and denote by M_1 the subset of M belonging to D. Since M is a continuum, M_1 is seen to contain infinitely many of the components C_x of $M-K$. Now let x' be a point of $M_1-(C_{x_1}+C_{x_2})$, and P a point of the arc x_1x_2 between x_1 and x_2. Since x' and P are both points of D, they are not separated by the continuum K. Moreover, since $x_1x_2-(x_1+x_2)$ belongs to $S-M$ and $M-C_{x_i}$, $(i=1, 2)$, is connected in view of conclusion (α), there exists a complementary domain D_i of C_{x_i} containing the points P and x', and hence the continuum C_{x_i}, $(i=1, 2)$, does not separate x' and P. It follows, therefore, that $K+C_{x_1}+C_{x_2}$ does not separate† x' and P, and hence there exists a simple continuous arc $x'P$ belong-

* See B. Knaster and C. Kuratowski, *Sur les ensembles connexes*, Fundamenta Mathematicae, vol. 2 (1921), pp. 206–255, in particular, p. 214, Theorem X.

† See B. Knaster and C. Kuratowski, *Sur les continus non-bornés*, Fundamenta Mathematicae, vol. 5 (1924), pp. 23–58, in particular, p. 35.

ing to $S-(K+C_{x_1}+C_{x_2})$. Let O denote the first point of the arc $x'P$ belonging to the arc x_1x_2, and x_3 the last point of the arc $x'O$ belonging to M. The point O is seen to be between x_1 and x_2 on the arc x_1x_2, and x_3 is a point of $M-K$ such that C_{x_3} is distinct from C_{x_1} and C_{x_2}.

Now let H_j, $(j=1, 2, 3)$, denote the continuum consisting of the component C_{x_j} and the arc Ox_j. Each of the three unbounded continua H_j contains the point O, no two of them have in common any point except O, and no one of them is separated by the omission of O. It then follows* that there is no complementary domain of the continuum $H=H_1+H_2+H_3$ whose boundary contains a point, distinct from O, of each of the three continua H_j. If $\mathfrak{D}$ denotes the complementary domain of H containing the continuum K, the boundary of $\mathfrak{D}$, aside from the point O, belongs to some two of the continua H_j. For definiteness, suppose that the boundary of $\mathfrak{D}$ is a subset of H_1+H_2. Let $\mathfrak{D}_0$ denote the complementary domain of the continuum H_1+H_2 containing the continuum C_{x_3}, and denote by M_0 the subset of M belonging to $\mathfrak{D}_0$. Since $\mathfrak{D}$ is a complementary domain of H, the domain $\mathfrak{D}_0$ is distinct from the domain $\mathfrak{D}$ and the point set M_0 does not contain K. Moreover, since M is a continuum, infinitely many of the components C_x belong to M_0.

Now consider the closed point set $N=M_0+C_{x_1}+C_{x_2}$. If N is connected, we have a contradiction to the assumption that C_{x_j}, $(j=1, 2, 3)$, is a component of $M-K$. Suppose, on the other hand, that N is not connected. It will now be shown that under this hypothesis the point set N is the sum of two mutually exclusive closed point sets N_1 and N_2 such that N_1 contains both C_{x_1} and C_{x_2}. Since N is closed, the assumption that N is not connected implies that $N=N_{10}+N_{20}$, where N_{10} and N_{20} are mutually exclusive closed point sets. It may be supposed without

* For let A be a point of $S-H$ and subject the plane to an inversion about A. If $\overline{O}$, $\overline{H}_j$, $(j=1, 2, 3)$, denote the images of O, H_j, $(j=1, 2, 3)$, under this inversion, then the bounded continua $L_j=\overline{H}_j+A$ satisfy the following conditions: each of the continua L_j contains the distinct points A and $\overline{O}$, no two of them have in common any point except A and $\overline{O}$, and no one of them is separated by the omission of $A+\overline{O}$. It then follows [see, for example, Moore, p. 291] that there is no complementary domain of the continuum $L=L_1+L_2+L_3$ whose boundary contains a point, distinct from A and $\overline{O}$, of each of the three continua L_j, $(j=1, 2, 3)$. Since each complementary domain of L is the image of a complementary domain of H, the proof of the above stated result is complete.

loss of generality that N_{10} contains more than one of the components C_x of $M-K$ which belongs to N. If N_{10} contains both C_{x_1} and C_{x_2}, the property stated above is true for $N_1=N_{10}$, $N_2=N_{20}$; if N_{10} contains neither C_{x_1} nor C_{x_2}, the property is true for $N_1=N_{20}$, $N_2=N_{10}$. Finally, suppose that (i, j) is a permutation of $(1, 2)$ such that N_{10} contains C_{x_i} and N_{20} contains C_{x_j}. Since N_{10} contains components of $M-K$ distinct from C_{x_i}, the closed point set N_{10} is not connected. Therefore, $N_{10}=N_{101}+N_{102}$, where N_{101} and N_{102} are mutually exclusive closed point sets and N_{101} contains C_{x_i}. The above stated property is then true for $N_1=N_{101}+N_{20}$, $N_2=N_{102}$. Now since N_1 contains both C_{x_1} and C_{x_2}, we see that N_2 is a subset of M_0 and hence contains no limit point of $M-N$. Consequently, N_2 and $M-N_2$ are mutually separated, which is a contradiction to the assumption that M is a continuum.

On the assumption that conclusion (β) of Lemma 1 is false we have thus been led to a contradiction. Hence this conclusion is true, and Lemma 1 is established.

4. *Remarks.* In a letter to me, J. H. Roberts has stated that he has discovered an example of a continuum in three-dimensional euclidean space showing that the result of the theorem of this paper does not hold when the condition that the continuum M be a plane continuum is omitted. There still remains the interesting question as to whether or not the statement obtained by replacing in our theorem the word "component" by "maximal strongly connected subset"* is true. That this latter question may also be answered in the negative when the condition that the continuum M be a plane continuum is omitted is a consequence of an example given by Knaster and Kuratowski of a three-dimensional indecomposable continuum each of whose composants is a continuum.†

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* A point set N is said to be *strongly connected* if for every two points x and y of N there exists a continuum which contains both x and y and which is a subset of N. A maximal strongly connected subset of N is a strongly connected subset of N which is not a proper subset of any other strongly connected subset of N.

† See B. Knaster and C. Kuratowski, *Sur les continus non-bornés*, Fundamenta Mathematicae, vol. 5 (1924), pp. 23–58, §4.

THE CONSTANTS IN WARING'S PROBLEM FOR ODD POWERS

BY R. D. JAMES

1. *Introduction.* In a previous paper* the writer determined the constants in the Hardy-Littlewood analysis of Waring's problem. It was then possible to obtain new universal theorems. The proof depended on the following results.

THEOREM A. (Theorem 46, page 429, and (C), page 442 in paper T.) *Let $k \geqq 6$ be an integer; s an integer; $a = 1/k$;*

$$d = [\log (k-1)/\log 2]; \quad D = (d+2)(k-1) - 2^{d+1} + 0.1;$$

$$s_2 = 4 + \zeta_k = 4 + \left[\frac{(k-2)\log 2 - \log k + \log (k-2)}{\log k - \log (k-1)}\right];$$

$$\eta = \frac{D(s-2) + k2^{k-2}(1 + (1-a)^{s_2-2})}{(s-2) - (k-2)2^{k-2} - k}.$$

Then every integer $n > C$, where $\log_e C = 20k^3 2^k$, is a sum of $s + s_2$ kth powers.

The proof of universal theorems was greatly simplified by Dickson's new method. Using Theorem A and very short tables he proved that† *every* integer is a sum of 259 seventh powers, 575 eighth powers, 981 ninth powers, 10711 twelfth powers, · · · . All these results are considerably better than those obtained by algebraic methods.

In another paper‡ the writer obtained new asymptotic results for odd powers. It is the purpose of the present note to point out that the constants of this paper may also be evaluated by the methods of paper T. This leads to still better universal

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* Transactions of this Society, vol. 36 (1934), pp. 395–444. This will be referred to as paper T.

† This Bulletin, vol. 39 (1933), Theorems 15–18, pp. 713–714.

‡ Proceedings of the London Mathematical Society, (2), vol. 37 (1934), pp. 257–291. This will be referred to as paper P.

theorems for odd powers. Corresponding to Theorem A we have the following theorem.

THEOREM B. *Let $k \geqq 7$ be an odd integer; $t \leqq k2^{k-2}$ an integer; $a=1/k$; $b=1/(k-1)$;*

$$d_a=[\log (k-1)/\log 2]; \quad d_b=[\log (k-2)/\log 2];$$

$$D_a=(d_a+2)(k-1)-2^{d_a+1}+0.1; \quad D_b=(d_b+2)(k-2)-2^{d_b+1}+0.1;$$

$s_2=4+\zeta_k$, *where ζ_k was defined in Theorem* A;

$$\eta=\frac{D_a(2k-1)(k-1)+D_bk(2t-4)+2^{k-2}k(k-1)(1+(1-a)^{s_2-2})}{(2k-1)(k-1)+k(2t-4)-2^{k-2}k(k-3)-k(k-1)};$$

$v=4$ when k is divisible by 15, $v=2$ in all other cases. Then every integer $n>C'$, where $\log_e C'=25k^2 2^v$, is a sum of $2t+s_2+2v+2k-1$ kth powers.

The use of this theorem instead of Theorem A enabled Dickson* to prove that *every* integer is a sum of 4425 eleventh powers. The Hardy-Littlewood result showed only that every *sufficiently large* integer was a sum of 4687 eleventh powers.

2. *Changes Required for C_{100}.* In order to obtain Theorem B there is one change which must be made in paper P. The constant C_{99} which occurs in the proof of Theorem 3 (page 289) is evaluated in precisely the same way as the constant C in paper T, and we shall not repeat the discussion. However, in Lemma 33 (page 290) the constant C_{100} depends on a prime number theory constant which has not been evaluated. Accordingly we replace Lemma 33 by the following lemma.

LEMMA A. *Define k, t, and v as in Theorem* B. *Let*

$p=3$ *when k is prime to* 6,

$p=5$ *when k is divisible by* 3 *and prime to* 10,

$p=2$ *in all other cases.*

Then given any integer $w>p^{(k+6)k^2}$ there exist $2v$ kth powers $w_1, w_2, \cdots, w_{2v}$ and integers $\mu \geqq 2$ and W such that

$$(1) \qquad w - 2tp^{(k+2+\mu)k^2} - \sum_{\alpha=1}^{2v} w_\alpha^k = p^{(k+2+\mu)k^2}W,$$

* Journal of the London Mathematical Society, vol. 9 (1934), pp. 201–206.

where

$$(2^k - 4)p^{(k+2+\mu)(k^2-k)} \leqq W \leqq (2^k - 2)p^{(k+2+\mu)(k^2-k)}.$$

The integer W satisfies the inequalities (8·32), for the value $N = p^{(k+2+\mu)k}$, on page 288 of paper P. For by (2·31), page 258, we have

$$\begin{aligned} 2t(P(M) + 1) + N^{k-1} &\leqq 2t(P(BN) + 1) + N^{k-1} \\ &= 2t\{(1 + B)^k + (1 - B)^k - 2\}N^{k-1} + 2t + N^{k-1} \\ &< (2^k - 6)N^{k-1} + 2t + N^{k-1} < (2^k - 4)N^{k-1}, \end{aligned}$$

since we have $2t \leqq k2^k < 2^{2k} < p^{(k+2+\mu)(k^2-k)} = N^{k-1}$.

3. *Evaluation of C'.* We shall now prove Lemma A and then show that the constant C' in Theorem B is equal to

$$vp^{(k+2)k^2} + (2t + 2^k)(2^k - 4)^{-1}p^{k^2}C_{99}^{k/(k-1)}. \tag{2}$$

Knowing $\log_e C_{99}$ (evaluated by the methods of paper T) we can therefore find $\log_e C'$. The result is $\log_e C' < 25k^3 2^{7^k}$.

To prove Lemma A, we require two additional lemmas.

LEMMA B. *Let $m(\geqq 1)$ and E be integers; $P \geqq 0$, $Q \geqq 0$, and $P + (m-1)^k \leqq E \leqq Q$. Then there exist at least m consecutive integers $i_\beta \geqq 0$, $(\beta = 1, 2, \cdots, m)$, which satisfy the inequalities*

$$P \leqq E - i_\beta^k \leqq P + kmQ^{(k-1)/k}. \tag{3}$$

PROOF. Let $\delta = (E - P)^{1/k}$. Then $[\delta] \geqq m - 1$, and we have

$$\begin{aligned} P &= E - \delta^k \leqq E - [\delta]^k \leqq E - ([\delta] - m + 1)^k \\ &= P + \delta^k - ([\delta] - m + 1)^k \leqq P + k(\delta - [\delta] + m - 1)\delta^{k-1} \\ &\leqq P + km\delta^{k-1} = P + km(E - P)^{(k-1)/k} \\ &\leqq P + kmQ^{(k-1)/k}. \end{aligned}$$

The integers $[\delta] - m + 1$, $[\delta] - m + 2$, $\cdots$, $[\delta]$ are all $\geqq 0$ and satisfy the inequalities (3).

LEMMA C. *Define v and p as before. Let $e \geqq 2$ and F be integers; $P \geqq 0$, $Q \geqq 0$ and*

$$P + vp^{ek} \leqq F \leqq Q. \tag{4}$$

Then there exist integers $h_1, h_2, \cdots, h_v$, and W such that

$$F = \sum_{\alpha=1}^{v} h_\alpha^k + p^e W,$$

where

$$Pp^{-e} \leqq W \leqq Pp^{-e} + kQ^{(k-1)/k}.$$

PROOF. It is known* that there is a solution of the congruence

$$F \equiv \sum_{\alpha=1}^{v} h_\alpha^k \pmod{p^e}$$

for v and p as defined. Hence there is an integer W such that

$$F = \sum_{\alpha=1}^{v} h_\alpha^k + p^e W. \tag{5}$$

For $\alpha = 1, 2, \cdots, v-1$ we choose h_α so that $0 \leqq h_\alpha < p^e$. Then from (4), (5) we have $P + p^{ek} \leqq F - \sum_{\alpha=1}^{v-1} h_\alpha^k \leqq Q$. In Lemma B with $m = p^e$, $E = F - \sum_{\alpha=1}^{v-1} h_\alpha^k$, one of the p^e consecutive integers i_β must be congruent to h_v modulo p^e, and we take h_v equal to this particular i_β. It follows from Lemma B that

$$P \leqq F - \sum_{\alpha=1}^{v} h_\alpha^k \leqq P + kp^e Q^{(k-1)/k},$$

and then from (5) that

$$Pp^{-e} \leqq W \leqq Pp^{-e} + kQ^{(k-1)/k}.$$

The proof of Lemma A now proceeds as follows: Let

$$Q(\mu) = (2t + 2^k)p^{(k+2+\mu)k^2} + vp^{(k+2)k^2}. \tag{6}$$

Since $w > p^{(k+6)k^2}$ and $2t \leqq k2^k$, we have

$$\begin{aligned} w &> p^{(k+4)k^2} \cdot p^{2k^2} > (k2^k + 2^k)p^{(k+4)k^2} + 4p^{(k+2)k^2} \\ &\geqq (2t + 2^k)p^{(k+4)k^2} + vp^{(k+2)k^2} = Q(2). \end{aligned}$$

Thus for every integer $w > p^{(k+6)k^2}$ there is an integer $\mu \geqq 2$ such that

$$Q(\mu) \leqq w < Q(\mu + 1). \tag{7}$$

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* See E. Landau, *Vorlesungen über Zahlentheorie*, vol. 1, 1927, Theorems 291 and 301.

Now

$$\begin{aligned} Q(\mu+1) - 2tp^{(k+2+\mu)k^2} &= (2t+2^k)p^{(k+3+\mu)k^2} + vp^{(k+2)k} \\ &\quad - 2tp^{(k+2+\mu)k} < (2t+2^k)p^{(k+3+\mu)k^2} \\ &\leqq (k2^k+2^k)p^{(k+3+\mu)k^2} < (2/k)^{k/(k-1)}p^{k(k+1)(k+2)+\mu k^2}. \end{aligned}$$

Hence we have

$$\begin{aligned} Q(\mu) - 2tp^{(k+2+\mu)k^2} &\leqq w - 2tp^{(k+2+\mu)k^2} \\ &< Q(\mu+1) - 2tp^{(k+2+\mu)k^2}, \\ 2^k p^{(k+2+\mu)k^2} + vp^{(k+2)k^2} &\leqq w - 2tp^{(k+2+\mu)k^2} \\ &< (2/k)^{k/(k-1)}p^{k(k+1)(k+2)+\mu k^2}. \end{aligned} \tag{8}$$

As in the proof of Lemma C, with $F = w - 2tp^{(k+2+\mu)k^2}$, $e = (k+2)k$, there is a solution of

$$w - 2tp^{(k+2+\mu)k^2} \equiv \sum_{\alpha=1}^{v} w_\alpha^k \pmod{p^{(k+2)k}},$$

with $0 \leqq w_\alpha < p^{(k+2)k}$. Hence

$$w - 2tp^{(k+2+\mu)k^2} = \sum_{\alpha=1}^{v} w_\alpha^k + p^{(k+2)k}H, \tag{9}$$

where from (8) we have

$$\begin{aligned} 2^k p^{(k+2+\mu)k^2} &\leqq p^{(k+2)k}H < (2/k)^{k/(k-1)}p^{k(k+1)(k+2)+\mu k^2}, \\ 2^k p^{(k+2)(k^2-k)+\mu k^2} &\leqq H < (2/k)^{k/(k-1)}p^{(k+2+\mu)k^2}. \end{aligned}$$

Since

$$2^k p^{(k+2)(k^2-k)+\mu k^2} > (2^k - 4)p^{(k+2)(k^2-k)+\mu k^2} + vp^{\mu k^2},$$

this gives

$$(2^k - 4)p^{(k+2)(k^2-k)+\mu k^2} + vp^{\mu k^2} \leqq H < (2/k)^{k/(k-1)}p^{(k+2+\mu)k^2}.$$

Next, by Lemma C, with $F = H$, $e = \mu k$,

$$P = (2^k - 4)p^{(k+2)(k^2-k)+\mu k^2}, \qquad Q = (2/k)^{k/(k-1)}p^{(k+2+\mu)k^2},$$

there exist integers $h_1, h_2, \cdots, h_v$, and W such that

$$H = \sum_{\alpha=1}^{v} h_\alpha^k + p^{\mu k} W, \tag{10}$$

where

$$\begin{aligned} (2^k - 4)p^{(k+2+\mu)(k^2-k)} \leqq W \leqq (2^k - 4)p^{(k+2+\mu)(k^2-k)} \\ + k(2/k)p^{(k+2+\mu)(k^2-k)}, \\ (2^k - 4)p^{(k+2+\mu)(k^2-k)} \leqq W \leqq (2^k - 2)p^{(k+2+\mu)(k^2-k)}. \end{aligned} \tag{11}$$

Hence from (9) and (10) we obtain

$$w - 2tp^{(k+2+\mu)k^2} - \sum_{\alpha=1}^{v} w_\alpha^k - \sum_{\alpha=1}^{v} (p^{k+2}h_\alpha)^k = p^{(k+2+\mu)k^2} W,$$

where W satisfies the inequalities (11). This proves Lemma A.

It remains to show how the constant C' of Theorem B is obtained. Proceeding as in §9 of paper P with $s = 2k-1$, $s_1 = 4+\zeta_k = s_2$, but using Lemma A instead of Lemma 33, we prove that w is a sum of

$$(2t + s_2 + 2v + 2k - 1)$$

kth powers, provided that $W > C_{99}$. We shall have $W > C_{99}$ if μ is chosen so that

$$(2^k - 4)p^{(k+2+\mu)k^2} > C_{99}^{k/(k-1)}.$$

From (6), (7) this inequality holds if

$$w > vp^{(k+2)k^2} + (2t + 2^k)(2^k - 4)^{-1} p^{k^2} C_{99}^{k/(k-1)},$$

which proves (2).

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GENERAL SOLUTION OF THE PROBLEM OF ELASTOSTATICS OF AN n-DIMENSIONAL HOMOGENEOUS ISOTROPIC SOLID IN AN n-DIMENSIONAL SPACE

BY H. M. WESTERGAARD

1. *Introduction.* Dealing with the important case of a three-dimensional solid subject to constant body forces (such as gravity) B. Galerkin* expressed the stresses and the displacements in terms of three functions, governed by the fourth-order equation $\Delta\Delta f = \text{const.}$, and mutually independent except through the boundary conditions. He has demonstrated the fruitfulness of his method in later papers.†

It is profitable to interpret Galerkin's three functions as components of a vector. Simplicity is gained and significance is added by doing this. It is proposed to call this vector *the Galerkin vector.* Its nature is such that only a slight amount of complexity is added in the general derivations by considering an n-dimensional space.

2. *Notation.* Let the following notation be used.

$i_1, i_2, \cdots, i_m, \cdots, i_p, \cdots, i_n$ = unit vectors in n directions perpendicular to one another; $m \neq p$.

$\boldsymbol{R} = i_1x_1 + i_2x_2 + \cdots + i_nx_n$ = radius vector drawn from the origin to any point; the point is called point $\boldsymbol{R}$.

$\boldsymbol{\varrho} = i_1\xi_1 + i_2\xi_2 + \cdots + i_n\xi_n$ = displacement = increment of $\boldsymbol{R}$. The point $\boldsymbol{R}$ moves to the position $\boldsymbol{R}+\boldsymbol{\varrho}$; $\boldsymbol{\varrho}$ is assumed small.

$\boldsymbol{P} = i_1P_1 + i_2P_2 + \cdots + i_nP_n$ = force.

$\boldsymbol{K} = i_1K_1 + i_2K_2 + \cdots + i_nK_n$ = body force which is distrib-

* B. Galerkin, *Contribution à la solution générale du problème de la théorie de l'élasticité dans le cas de trois dimensions,* Comptes Rendus, vol. 190 (1930), p. 1047; *Contribution à l'investigation des tensions et des déformations d'un corps élastique isotrope* (in Russian), Comptes Rendus de l'Académie des Sciences de l'URSS, (1930), p. 353.

† Comptes Rendus, vol. 193 (1931), p. 568; vol. 194 (1932), p. 1440; vol. 195 (1932), p. 858; and papers in Russian: Comptes Rendus de l'Académie des Sciences de l'URSS, (1931), p. 273 and p. 281; Messenger of Mechanics and Applied Mathematics, Leningrad, vol. 1 (1931), p. 49; Transactions of the Scientific Research Institute of Hydrotechnics, vol. 10 (1933), p. 5.

uted through the solid, per unit of magnitude of the region considered; of the dimension force times distance $^{-n}$; a function of $\boldsymbol{R}$.

S_m = section defined by x_m = const. The section is called a front face if the part of the solid dealt with lies on the side of smaller values of x_m; and a back face if the part dealt with lies on the side of greater values of x_m.

$\mathbf{s}_m$ = vector representing internal force or stress on a small region of the front face S_m at point $\boldsymbol{R}$, per unit of magnitude of this region; of the dimension force times distance^{-n+1}; and $-\mathbf{s}_m$ = stress on the same region of the back face S_m.

σ_m = component of $\mathbf{s}_m$ in the direction i_m; normal stress in the direction of i_m.

τ_{mp} = component of $\mathbf{s}_m$ in the direction of i_p, $(p \neq m)$; shearing stress on the front face S_m in the direction i_p.

$\Theta = \sigma_1 + \sigma_2 + \cdots + \sigma_n$ = bulk stress.

E = Young's modulus of elasticity (equation (6)).

G = modulus of elasticity in shear (equation (7)).

μ = Poisson's ratio of lateral contraction (equation (6)).

$\boldsymbol{F} = i_1X_1 + i_2X_2 + \cdots + i_nX_n$ = Galerkin vector.

$$\nabla = i_1\frac{\partial}{\partial x_1} + i_2\frac{\partial}{\partial x_2} + \cdots + i_n\frac{\partial}{\partial x_n}.$$

$$\operatorname{div}\boldsymbol{F} = \frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \cdots + \frac{\partial X_n}{\partial x_n}.$$

$$\Delta = \operatorname{div}\nabla = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

3. *Equilibrium.* A set of forces $\boldsymbol{P}$ applied at the points $\boldsymbol{R}$ is in equilibrium when the n conditions

$$\sum P_m = 0, \tag{1}$$

and the $n(n-1)/2$ conditions

$$\sum P_m(x_p - C_p) = \sum P_p(x_m - C_m) \tag{2}$$

are satisfied, each sum including all the force components in the single direction of either i_m or i_p, C_m and C_p being constants which are interpreted as coordinates in the directions of i_m and i_p of an arbitrary fixed point C.

Consider an element of the solid bounded by back faces $S_1, S_2, \cdots, S_n$ intersecting at the point $\boldsymbol{R}$, and by front faces

$S_1, S_2, \cdots, S_n$ intersecting at the point $R+dR$. With all the forces included which act on this element, and with C at the center of the element, equation (2) leads to the $n(n-1)/2$ conditions

$$\tau_{mp} = \tau_{pm}. \tag{3}$$

Equation (1) leads to the n conditions

$$\frac{\partial \sigma_m}{\partial x_m} + \sum_{\substack{1,2,\cdots,n \\ \text{excl. } m}}^{p} \frac{\partial \tau_{pm}}{\partial x_p} + K_m = 0. \tag{4}$$

Because of equation (3), equation (4) may be rewritten in the form

$$\operatorname{div} s_m + K_m = 0. \tag{5}$$

4. *Elasticity.* Hooke's law of stresses and deformations is stated in the following form. The strain in the direction of i_m is

$$\frac{\partial \xi_m}{\partial x_m} = \frac{1+\mu}{E}\sigma_m - \frac{\mu}{E}\Theta, \tag{6}$$

and the detrusion in the directions of i_m and i_p is

$$\frac{\partial \xi_m}{\partial x_p} + \frac{\partial \xi_p}{\partial x_m} = \frac{\tau_{mp}}{G}. \tag{7}$$

With the constants E and μ given, only one value of G leads to isotropy. The value

$$G = \frac{E}{2(1+\mu)} \tag{8}$$

represents isotropy in the cases $n=2$ and $n=3$, and is assigned to G here. That this value actually represents isotropy for any value of n is concluded from the form of equation (13), which is derived from equations (5) to (8). Equation (6) gives

$$\operatorname{div} \varrho = \frac{1-(n-1)\mu}{E}\Theta, \tag{9}$$

which, with (8), permits the rewriting of (6) in the form

$$\sigma_m = 2G\left(\frac{\partial \xi_m}{\partial x_m} + \frac{\mu}{1-(n-1)\mu}\operatorname{div} \varrho\right). \tag{10}$$

Equations (7) and (10) lead to the following formula for the resultant stress on the front face S_m:

$$s_m = G\left(\nabla\xi_m + \frac{\partial \varrho}{\partial x_m}\right) + i_m \frac{2G\mu}{1-(n-1)\mu} \operatorname{div} \varrho. \tag{11}$$

Substitution from equation (11) in equation (5) gives

$$\Delta\xi_m + \frac{1-(n-3)\mu}{1-(n-1)\mu} \frac{\partial}{\partial x_m} \operatorname{div} \varrho + \frac{K_m}{G} = 0, \tag{12}$$

or, in vector form,

$$\left(\Delta + \frac{1-(n-3)\mu}{1-(n-1)\mu} \nabla \operatorname{div}\right)\varrho + \frac{K}{G} = 0. \tag{13}$$

Isotropy is attained because equation (13) is independent of the orientation of the axes of coordinates.

5. *The Galerkin Vector.* The general solution of equation (13) is the general solution of the problem. A difficulty arises from the interdependence of the components. This difficulty is overcome by introducing the Galerkin vector $\boldsymbol{F}$.

The displacement is expressed as

$$\varrho = \frac{1}{2G}(c\Delta - \nabla \operatorname{div})\boldsymbol{F} \tag{14}$$

in which c is a constant yet to be selected. By substituting ϱ from equation (14) in equation (13) and noting that

$$\Delta \nabla \operatorname{div} = \nabla \operatorname{div} \Delta = \nabla \operatorname{div} \nabla \operatorname{div}, \tag{15}$$

it is found that the terms containing the combined operators shown in equations (15) disappear when

$$c = \frac{2(1-(n-2)\mu)}{1-(n-3)\mu}. \tag{16}$$

Then equations (14) and (13) become

$$\varrho = \frac{1}{2G}\left[\frac{2(1-(n-2)\mu)}{1-(n-3)\mu}\Delta - \nabla \operatorname{div}\right]\boldsymbol{F}, \tag{17}$$

$$\Delta^2 \boldsymbol{F} = -\frac{1-(n-3)\mu}{1-(n-2)\mu}\boldsymbol{K}. \tag{18}$$

The general solution of equation (18) defines the general solution for ϱ through equation (17).

6. *Stresses.* The vector F and its components $X_1, X_2, \cdots, X_n$ lend themselves to expressions for the stresses. Equation (17) gives

$$\xi_m = \frac{1}{2G}\left[\frac{2(1-(n-2)\mu)}{1-(n-3)\mu}\Delta X_m - \frac{\partial}{\partial x_m}\operatorname{div} F\right], \tag{19}$$

and

$$\operatorname{div}\varrho = \frac{1-(n-1)\mu}{2(1-(n-3)\mu)G}\Delta \operatorname{div} F. \tag{20}$$

Substitution from equations (19) and (20) in equations (10), (7), (9), and (11) leads to the formulas

$$\sigma_m = \frac{2(1-(n-2)\mu)}{1-(n-3)\mu}\frac{\partial \Delta X_m}{\partial x_m} + \left[\frac{\mu}{1-(n-3)\mu}\Delta - \frac{\partial^2}{\partial x_m^2}\right]\operatorname{div} F, \tag{21}$$

$$\tau_{mp} = \frac{1-(n-2)\mu}{1-(n-3)\mu}\left[\frac{\partial \Delta X_m}{\partial x_p} + \frac{\partial \Delta X_p}{\partial x_m}\right] - \frac{\partial^2}{\partial x_m \partial x_p}\operatorname{div} F, \tag{22}$$

$$\Theta = \frac{1+\mu}{1-(n-3)\mu}\Delta \operatorname{div} F, \tag{23}$$

and

$$s_m = \frac{1-(n-2)\mu}{1-(n-3)\mu}\left[\nabla \Delta X_m + \frac{\partial}{\partial x_m}\Delta F\right] + \left[\frac{i_m \mu}{1-(n-3)\mu}\Delta - \frac{\partial}{\partial x_m}\nabla\right]\operatorname{div} F. \tag{24}$$

The form of equations (17) and (18) shows that F is independent of the orientation of the axes of coordinates. It follows that equations (19) to (24) continue to apply with the same F after a re-orientation of the axes of coordinates.

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SOME BESSEL FUNCTION EXPANSIONS

BY W. H. WISE

This is a formal derivation of some asymptotic Bessel function expansions. An interesting feature of the derivation is that it makes use of Heaviside's *generalized exponential series*.* The chief results arrived at, the Hankel function expansions (4) and (6), are generalizations of formulas (5) and (8) on page 141 in Watson's *Theory of Bessel Functions* in the same sense that Heaviside's generalized exponential series is a generalization of the ordinary exponential series.

In the course of some work on dipole radiation the author stumbled onto a particular case of the following relation

$$(1) \qquad H_0^{(1)}(z(1-a)^{1/2}) = \sum_{n=-\infty}^{n=\infty} \frac{(az/2)^{n-\nu}}{\Pi(n-\nu)} H_{n-\nu}^{(1)}(z).$$

In the particular case where ν is $1/2$, a is in the fourth quadrant and z is in the first quadrant and has a large real part. Except for a missing common factor, the left-hand side in the particular case is half the Zenneck wave component of the wave function (as measured at the surface of the earth) for a vertical dipole centered on the interface between air and earth, the descending half of the series is the negative of the sky wave component of the wave function and the ascending part of the series is the convergent expansion for half the Zenneck wave component plus the sky wave component.

The suggestion immediately offered itself that (1) should hold for all values of ν, not just for $\nu = 1/2$.

If we assume for the moment that (1) holds for all ν, and differentiate with respect to z, we get

$$H_1^{(1)}(z(1-a)^{1/2}) = \frac{-1}{(1-a)^{1/2}} \sum_{n=-\infty}^{n=\infty} \frac{(az/2)^{n-\nu}}{\Pi(n-\nu)} H_{n-\nu-1}^{(1)}(z)$$

$$= \frac{-1}{(1-a)^{1/2}} \sum_{n=-\infty}^{n=\infty} \frac{(az/2)^{n-\nu}}{\Pi(n-\nu)} \left[\frac{2(n-\nu)}{z} H_{n-\nu}^{(1)}(z) - H_{n-\nu+1}^{(1)}(z) \right]$$

* See John R. Carson, Transactions of this Society, vol. 31 (1929), pp. 782–792.

or

$$(2)\quad H_1^{(1)}(z(1-a)^{1/2}) = (1-a)^{1/2} \sum_{n=-\infty}^{n=\infty} \frac{(az/2)^{n-\nu}}{\Pi(n-\nu)} H_{n-\nu+1}^{(1)}(z).$$

By use of the recurrence relation for Bessel functions,

$$Z_{p+1}(z) = \frac{2p}{z} Z_p(z) - Z_{p-1}(z),$$

we get from (1) and (2)

$$(3)\quad \begin{aligned} H_2^{(1)}(z(1-a)^{1/2}) &= \sum_{n=-\infty}^{n=\infty} \frac{(az/2)^{n-\nu}}{\Pi(n-\nu)} \left[\frac{2}{z} H_{n-\nu+1}^{(1)}(z) - H_{n-\nu}^{(1)}(z) \right] \\ &= \sum_{n=-\infty}^{n=\infty} \frac{(az/2)^{n-\nu}}{\Pi(n-\nu)} \left[H_{n-\nu+2}^{(1)}(z) - (n-\nu)\frac{2}{z} H_{n-\nu+1}^{(1)}(z) \right] \\ &= \sum_{n=-\infty}^{n=\infty} \frac{(az/2)^{n-\nu}}{\Pi(n-\nu)} H_{n-\nu+2}^{(1)}(z) \\ &\qquad - a \sum_{n=-\infty}^{n=\infty} \frac{(az/2)^{n-\nu-1}}{\Pi(n-\nu-1)} H_{n-\nu+1}^{(1)}(z) \\ &= (1-a) \sum_{n=-\infty}^{n=\infty} \frac{(az/2)^{n-\nu}}{\Pi(n-\nu)} H_{n-\nu+2}^{(1)}(z), \end{aligned}$$

and it is now more or less evident that in general

$$(4)\quad H_m^{(1)}(z(1-a)^{1/2}) = (1-a)^{m/2} \sum_{n=-\infty}^{n=\infty} \frac{(az/2)^{n-\nu}}{\Pi(n-\nu)} H_{n-\nu+m}^{(1)}(z).$$

Equation (4) is one of a pair of similar formulas. If, in place of (1), we start out with the idea that

$$(5)\quad H_0^{(1)}(z(1+a)^{1/2}) = \sum_{n=-\infty}^{n=\infty} \frac{(az/2)^{n-\nu}}{\Pi(n-\nu)} H_{-(n-\nu)}^{(1)}(z),$$

then steps identical with those taken in going from (1) to (4) lead us to

$$(6)\quad H_{-m}^{(1)}(z(1+a)^{1/2}) = (1+a)^{m/2} \sum_{n=-\infty}^{n=\infty} \frac{(az/2)^{n-\nu}}{\Pi(n-\nu)} H_{-(n-\nu+m)}^{(1)}(z),$$

for integral values of m.

On page 180 in Watson's *Theory of Bessel Functions* we find the relation

$$(7) \qquad H_p^{(1)}(z) = \frac{e^{-i(\pi/2)p}}{i\pi} \int_{-\infty - i\pi/2}^{\infty + i\pi/2} e^{iz\cosh w - pw} dw,$$

good for all values of p and for $|\arg z| < \pi/2$.

By setting (7) in the right-hand side of (6), we get

$$(a) \quad (1+a)^{m/2} \sum_{n=-\infty}^{n=\infty} \frac{(az/2)^{n-\nu}}{\Pi(n-\nu)} \frac{1}{i\pi} \int e^{iz\cosh w - (w+i\pi/2)(-n+\nu-m)} dw,$$

$$(b) \quad = (1+a)^{m/2} \frac{1}{i\pi} \int e^{iz\cosh w + m(w+i\pi/2)} \sum_{n=-\infty}^{n=\infty} \frac{\left(\frac{az}{2} e^{w+i\pi/2}\right)^{n-\nu}}{\Pi(n-\nu)} dw.$$

Now it is known that

$$\sum_{n=-\infty}^{n=\infty} \frac{z^{n-\nu}}{\Pi(n-\nu)}$$

is a formal series representation for e^z; it is Heaviside's generalized exponential series. So the series in the integrand can be summed. The integral now is

$$(1+a)^{m/2} \frac{1}{i\pi} \int e^{iz\cosh w + m(w+i\pi/2)} \exp\left(\frac{az}{2} e^{w+i\pi/2}\right) dw$$

$$(c) \quad = (1+a)^{m/2} \frac{1}{i\pi} \int e^{iz\cosh w} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{az}{2}\right)^n e^{(n+m)(w+i\pi/2)} dw$$

$$= (1+a)^{m/2} \sum_{n=0}^{\infty} \frac{(az/2)^n}{n!} H_{-m-n}^{(1)}(z) = H_{-m}^{(1)}(z(1+a)^{1/2}).$$

The right-hand side of (4) may be summed in the same way.

The series in (b) is a good representation for $\exp((az/2)e^{w+i\pi/2})$ over only a part of the integration range, but, since the series can be recognized where it is valid, its sum may be used where the series is not valid. Because the series in (b) is a good representation for $\exp((az/2)e^{w+i\pi/2})$ over only a part of the integration range the integral (b) is meaningless except as it is interpreted as (a) or (c). This formal summation of the right-hand sides of (4) and (6) is therefore not entirely satisfactory. The following argument will be more convincing and inciden-

tally indicates that by the application of sufficient ingenuity the steps from (a) to (b) and (b) to (c) could be justified.

Since the right-hand side of (8) is a solution of Bessel's equation it must be possible to determine constants A and B such that

$$\begin{aligned} &AH_m^{(1)}(z(1-a)^{1/2}) + BJ_m(z(1-a)^{1/2}) \\ &\quad = (1-a)^{m/2} \sum_{n=-\infty}^{n=\infty} \frac{(az/2)^{n-\nu}}{\Pi(n-\nu)} H_{n-\nu+m}^{(1)}(z). \end{aligned} \tag{8}$$

The right-hand side of (8) and the Bessel functions on the left satisfy the partial differential equation

$$2(1-a)\frac{\partial u}{\partial a} + z\frac{\partial u}{\partial z} = 0.$$

It follows that A and B are independent of a.

To see that $A=1$ and $B=0$, let $z\to\infty$ and $a\to 0$ while $-azi/2$ keeps some large constant value such that Heaviside's generalized exponential series is valid. Then each $H_p^{(1)}(z)$ may be replaced by the first term of the expansion

$$H_p^{(1)}(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} e^{i(z-(2p+1)\pi/4)} \left(1 + i\frac{4p^2-1}{1!8z} - \frac{(4p^2-1)(4p^2-3^2)}{2!8^2z^2} + \cdots\right),$$

and the principal terms in (8) are, after canceling the common factor $(2/(\pi z))^{1/2}$,

$$\begin{aligned} &Ae^{i[z(1-a/2)-(2m+1)\pi/4]} + B\cos\left[z\left(1-\frac{a}{2}\right) - \frac{2m+1}{4}\pi\right] \\ &\quad = \sum_{n=-\infty}^{n=\infty} \frac{(-azi/2)^{n-\nu}}{\Pi(n-\nu)} e^{i(z-(2m+1)\pi/4)} \\ &\quad = e^{i[z(1-a/2)-(2m+1)\pi/4]}. \end{aligned} \tag{9}$$

It follows that $A=1$ and $B=0$.

If the $H_{n-\nu+m}^{(1)}(z)$ in (8) are replaced by $J_{n-\nu+m}(z)$, then the right-hand side of (9) is replaced by

$$\sum_{n=-\infty}^{n=\infty} \frac{(az/2)^{n-\nu}}{\Pi(n-\nu)} \cos\left[z - \frac{2(n-\nu+m)+1}{4}\pi\right]$$

$$= \sum_{n=-\infty}^{n=\infty} \left\{ \frac{(-azi/2)^{n-\nu}}{\Pi(n-\nu)} \cdot \frac{e^{i(z-(2m+1)\pi/4)}}{2} + \frac{(azi/2)^{n-\nu}}{\Pi(n-\nu)} \cdot \frac{e^{-i(z-(2m+1)\pi/4)}}{2} \right\}$$

$$= \frac{1}{2} e^{i[z(1-a/2)-(2m+1)\pi/4]} + \frac{1}{2} e^{-i[z(1-a/2)-(2m+1)\pi/4]}$$

$$= \cos\left[z\left(1-\frac{a}{2}\right) - \frac{2m+1}{4}\pi\right],$$

and we must conclude that $A=0$ and $B=1$.

The differential equation satisfied by

$$y = H_m^{(1)}(z(1-a)^{1/2}) - (1-a)^{m/2} \sum_{n=-\infty}^{n=-N-1} \frac{(az/2)^{n-\nu}}{\Pi(n-\nu)} H_{n-\nu+m}^{(1)}(z)$$

$$= (1-a)^{m/2} \sum_{n=-N}^{n=\infty} \frac{(az/2)^{n-\nu}}{\Pi(n-\nu)} H_{n-\nu+m}^{(1)}(z)$$

is

$$y'' + \frac{y'}{z} + \left(1 - a - \frac{m^2}{z^2}\right) y$$

$$= a(1-a)^{m/2} \frac{(az/2)^{-N-\nu-1}}{\Pi(-N-1-\nu)} H_{-N-1-\nu+m}^{(1)}(z). \tag{10}$$

This differential equation has two particular integrals, $\sum_{-N}^{+\infty}$ and $-\sum_{-\infty}^{-N-1}$, and they differ by the complementary function $H_m^{(1)}(z(1-a)^{1/2})$. The divergent particular integral and the complementary function are inextricably tangled up in the convergent particular integral.

If we choose a sector of the complex plane where $H_p^{(1)}(z)$ vanishes as z goes to infinity we can see by inspection of the differential equation that as z goes to infinity the particular integral may behave in either one of two different ways. It may approach zero as does the right-hand side of the differential equation. If it does not approach zero as does the right-hand

side, it must be approaching a Bessel function in such a way that it does not completely satisfy Bessel's equation, but does so with a small and rapidly vanishing residue, the right-hand side of (10). The divergent particular integral must be the one that vanishes like the right-hand side. The convergent particular integral must be the one that eventually behaves like a Bessel function.

If $H^{(1)}$ is replaced by J the reasoning must be altered to specify that the imaginary part of z remains finite while the real part of z goes to infinity.

Formulas (4) and (6) are generalizations of formulas (5) and (8) on page 141 in Watson. The formulas in Watson are obtained by making ν zero, or any integer.

It should be noted that in verifying (4) and (6) it was not necessary to require that m be an integer. However, the restrictions on Watson's ν must apply to m also.

If in (4) we put $m=-1/2$ and replace $H^{(1)}$ by J, we get, since $\cos x=(\pi x/2)^{1/2}J_{-1/2}(x)$,

$$(11)\quad \left(\frac{2}{\pi z}\right)^{1/2}\cos\,(z(1-a)^{1/2})=\sum_{n=-\infty}^{n=\infty}\frac{(az/2)^{n-\nu}}{\Pi(n-\nu)}J_{n-\nu-1/2}(z).$$

If in (6) we put $m=-1/2$ and replace $H^{(1)}$ by J, we get, since $\sin x=(\pi x/2)^{1/2}J_{1/2}(x)$,

$$(12)\quad \left(\frac{2}{\pi z}\right)^{1/2}\sin\,(z(1+a)^{1/2})=\sum_{n=-\infty}^{n=\infty}\frac{(az/2)^{n-\nu}}{\Pi(n-\nu)}J_{-(n-\nu-1/2)}(z).$$

If we put $az/2=t$, then (11) and (12) are

$$(13)\quad \left(\frac{2}{\pi z}\right)^{1/2}\cos\,(z^2-2zt)^{1/2}=\sum_{n=-\infty}^{n=\infty}\frac{t^{n-\nu}}{\Pi(n-\nu)}J_{n-\nu-1/2}(z),$$

$$(14)\quad \left(\frac{2}{\pi z}\right)^{1/2}\sin\,(z^2+2zt)^{1/2}=\sum_{n=-\infty}^{n=\infty}\frac{t^{n-\nu}}{\Pi(n-\nu)}J_{-(n-\nu-1/2)}(z).$$

Equations (13) and (14) are generalizations of the Glaisher formulas (3) and (4) on page 140 in Watson. The Glaisher formulas are obtained by making ν zero, or any integer.

If in (11) we put $\nu=1/2$, we get

$$(15)\quad \left(\frac{2}{\pi z}\right)^{1/2}\cos\,(z(1-a)^{1/2})=\sum_{n=-\infty}^{n=\infty}\frac{(az/2)^{n-1/2}}{\Pi(n-1/2)}J_{n-1}(z)$$

or

$$a^{1/2}\cos\left(z(1-a)^{1/2}\right) = azJ_0(z) + \left(\frac{a^2z^2}{1\cdot 3} - 1\right)J_1(z)$$

$$+\left(\frac{a^3z^3}{1\cdot 3\cdot 5} - \frac{1}{az}\right)J_2(z) + \left(\frac{a^4z^4}{1\cdot 3\cdot 5\cdot 7} - \frac{1\cdot 3}{a^2z^2}\right)J_3(z) \tag{16}$$

$$+\left(\frac{a^5z^5}{1\cdot 3\cdot 5\cdot 7\cdot 9} - \frac{1\cdot 3\cdot 5}{a^3z^3}\right)J_4(z) + \cdots .$$

It will be observed that the left-hand side of (16) is an even function of z while the right-hand side is an odd function of (z). This merely indicates that $|\arg z| < \pi/2$ and is not surprising.

As a matter of curiosity (16) was computed with $z=10$ and $a=1$. The ascending part of the series, from J_0 to J_{15} inclusive, is 1.0668. The descending part of the series, from $-J_1$ on, is

$$-0.0435 \quad -0.0255 \quad -0.0018 \quad +0.0033 \quad +0.0025 \quad +0.0001$$
$$-0.0023 \quad -0.0043 \quad -0.0059 \quad -0.0072 \quad -0.0081 \quad -0.0087$$

and so on. The sum of the first seven terms is -0.0671, which is a better check than one would expect from these figures The decision as to where to terminate the asymptotic series is complicated by the two changes in sign.

If in (12) we put $\nu = 1/2$, we get

$$\left(\frac{2}{\pi z}\right)^{1/2}\sin\left(z(1+a)^{1/2}\right) = \sum_{n=-\infty}^{n=\infty}\frac{(az/2)^{n-1/2}}{\Pi(n-1/2)}J_{-n+1}(z), \tag{17}$$

or

$$a^{1/2}\sin\left(z(1+a)^{1/2}\right) = azJ_0(z) - \left(\frac{a^2z^2}{1\cdot 3} - 1\right)J_1(z)$$

$$+\left(\frac{a^3z^3}{1\cdot 3\cdot 5} - \frac{1}{az}\right)J_2(z) - \left(\frac{a^4z^4}{1\cdot 3\cdot 5\cdot 7} - \frac{1\cdot 3}{a^2z^2}\right)J_3(z) + \cdots . \tag{18}$$

With $z=10$ and $a=1$, and so using the terms computed for (16), we find that the left-hand side of (18) is 1.0000, the ascending part of the series is 0.9786, the asymptotic part is 0.0230, and the right side is 1.0016.

Since both sides of (18) are odd functions of z, this formula is only restricted to large absolute values of z.

BELL TELEPHONE LABORATORIES

ON APPROXIMATION BY POLYNOMIALS TO A FUNCTION ANALYTIC IN A SIMPLY CONNECTED REGION*

BY O. J. FARRELL

In a previous paper† the writer studied expansions in series of polynomials of a function $f(z)$ analytic in a limited simply connected region G where $f(z)$ is known either to be bounded in G or such that the double integral over G of the pth power $(p>0)$ of the modulus of $f(z)$ exists.‡ The present note contains an extension of each of the two theorems obtained in the earlier paper. The extended theorems now read as follows.

THEOREM A. *Let G be a limited simply connected region of the z plane. Then in order that corresponding to every function $f(z)$ analytic and bounded in G there shall exist a sequence of polynomials $\{p_n(z)\}$ which converge to $f(z)$ in G as $n\to\infty$ and at the same time such that*

$$\overline{\lim_{n\to\infty}}\,[\,|\,p_n(z)\,|,\ z \text{ in } G] \leqq \overline{\text{bound}}\,[\,|\,f(z)\,|,\ z \text{ in } G], \tag{1}$$

it is necessary and sufficient that the boundary of G be also the boundary of an infinite region.

THEOREM B. *In the z plane let G be a limited simply connected region whose boundary is also the boundary of an infinite region. Let $f(z)$ be analytic in G and such that*

$$\iint_G |\,f(z)\,|^p dS, \qquad (p>0), \tag{2}$$

exists. Then there exists a sequence of polynomials $\{p_n(z)\}$ such that

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* Presented to the Society, September 4, 1934.

† This Bulletin, vol. 40 (1934), pp. 908–914.

‡ The writer is indebted to Professor J. L. Walsh for having suggested a study of these two problems and also to Professor Torsten Carleman for sending a reprint of his paper on approximation to analytic functions by linear aggregates of prescribed powers (Arkiv för Matematik, Astronomi och Fysik, vol. 17 (1923), pp. 1–30).

$$\lim_{n\to\infty} \iint_G |f(z) - p_n(z)|^p dS = 0. \tag{3}$$

It will be noticed that in the first of these theorems we no longer say that there exist polynomials $\{p_n(z)\}$ which converge to $f(z)$ continuously in G, but merely that there exist polynomials $\{p_n(z)\}$ which converge to $f(z)$ in G. For if polynomials $\{p_n(z)\}$ converge to $f(z)$ in G so that (1) holds, these polynomials are uniformly bounded in G and thus form in G a normal family of analytic functions from which can be chosen a subsequence converging to $f(z)$ continuously in G. Hence, whenever there exists a sequence $\{p_n(z)\}$ converging to $f(z)$ in G so that (1) holds, there exists also a sequence which fulfills (1) and converges to $f(z)$ continuously in G. It will be seen too that in the second theorem we no longer say that there exist polynomials $\{p_n(z)\}$ which converge to $f(z)$ continuously in G and for which (3) holds, but merely that there exist polynomials $\{p_n(z)\}$ for which (3) holds. This is because we have since found in the literature a lemma by Walsh* giving assurance that if (3) holds, the polynomials $\{p_n(z)\}$ do converge to $f(z)$ continuously in G, so that specific mention of the convergence may be omitted.

The proof of Theorems A and B requires only a slight modification of the proof of the two corresponding theorems in the previous paper. This modification is brought about by observing that if G is any limited simply connected region whose boundary also bounds an infinite region, then there exists a sequence of regions $\{G_n\}$, each of which is a Jordan region lying interior to its predecessor and which are all such that the sequence $\{G_n\}$ converges to G as kernel.† If we use such a sequence of regions $\{G_n\}$, the proofs of Theorems 1 and 2 of the previous paper apply to Theorems A and B, respectively, of the present note. It is to be remarked, however, that uniform approximation to $f_n(z)$ or $F_n(z)$ in $\overline{G}$ by a polynomial with arbitrarily small error does not now follow directly from the theorem of Walsh that was

* Transactions of this Society, vol. 33 (1931), pp. 370–388, Lemma on p. 387.

† Compare Carathéodory, Mathematische Annalen, vol. 72 (1912), pp. 107–144, Chapter 3; or Walsh, Transactions of this Society, vol. 32 (1930), pp. 335–390, proof of Theorem X.

used before, but does follow indirectly from it, since $f_n(z)$ or $F_n(z)$, being analytic in the closed Jordan region $\overline{G}_{n+1}$, can be uniformly approximated with arbitrarily small error in $\overline{G}_{n+1}$ by a polynomial, and hence can be so approximated in $\overline{G}$. Indeed, Runge's classical theorem on polynomial approximation could be applied here and for that matter could have been used in the previous paper. For the closed region $\overline{G}$ is interior to every region G_n and hence the function $f_n(z)$ or $F_n(z)$ can be approximated as closely as desired in $\overline{G}$ by a polynomial in z.

The proof of Theorem B and the proof of the sufficiency of the condition of Theorem A are the same from this point on as for the corresponding theorems in the earlier paper. And the proof of the necessity of the condition of Theorem A is also contained there in §5 as "A Remark on Theorem 1."

The writer hopes in a forthcoming paper to be able to determine the most general type of limited simply connected region to which Theorem B can be extended. That *this theorem does not hold for an arbitrary finite simply connected region* is shown by the following simple example.

Let G be taken as the region bounded by the two circles $|z| = a$, $|z| = b$, $b > a$, and by the line segment $a \leqq z \leqq b$. Let $f(z) = 1/z$. Denote by Q the doubly connected region bounded by the two circles. If now there existed a positive number p together with a sequence of polynomials $\{p_n(z)\}$ such that

$$\lim_{n\to\infty} \iint_G |p_n(z) - 1/z|^p dS = 0,$$

it would follow that

$$\lim_{n\to\infty} \iint_Q |p_n(z) - 1/z|^p dS = 0. \tag{4}$$

Consequently the polynomials $\{p_n(z)\}$ would converge* to $1/z$ in Q and the convergence would be uniform on any closed point set lying in Q, say on a circle $|z| = c$, $a < c < b$. Hence, the polynomials $\{p_n(z)\}$ would converge uniformly on and within the circle $|z| = c$ to a limit function analytic within this circle. But

* This convergence would follow by the lemma of Walsh to which reference was made above in the paragraph immediately following the statement of Theorem B.

such a function could not be equal to $1/z$ in the ring region between the circles $|z| = a$ and $|z| = c$.

The main feature of this example is the use of the lemma of Walsh whereby we know that if in a finite region R we have a sequence of polynomials $\{p_n(z)\}$ for which

$$\lim_{n\to\infty} \iint_R |p_n(z) - f(z)|^p dS = 0, \qquad (p > 0),$$

where $f(z)$ is a given function analytic in R, then the polynomials $\{p_n(z)\}$ converge to $f(z)$ continuously in R. The converse is not always true, as was shown by an example in §4 of our previous paper. There is, however, a qualified form of converse which does hold for an arbitrary limited region and for an arbitrary function analytic therein. We may state this result as follows.

THEOREM C. *Let R be a limited region of the z plane, and let $f(z)$ be analytic in R and such that*

$$\iint_R |f(z)|^p dS, \qquad (p > 0),$$

exists. If polynomials $\{p_n(z)\}$ exist which converge to $f(z)$ continuously in R and for which

$$\lim_{n\to\infty} \iint_R |p_n(z)|^p dS = \iint_R |f(z)|^p dS,$$

then

$$\lim_{n\to\infty} \iint_R |f(z) - p_n(z)|^p dS = 0.$$

The proof of this theorem is already contained in the latter part of the proof of Theorem 2 in our previous paper.

We close this note with a result closely connected with Theorem A.

THEOREM D. *Let G denote a limited simply connected region whose boundary does not bound an infinite region. Let B denote the boundary of the infinite region among the regions into which the closed region $\overline{G}$ separates the plane, and let Γ denote the region (also simply connected) consisting of all the points which can be joined to an arbitrary fixed point of G by a Jordan arc containing*

no point of B. Let $f(z)$ be analytic and bounded in G. A necessary and sufficient condition for the existence of polynomials $\{p_n(z)\}$ which converge to $f(z)$ in G so that (1) *holds is that there exist a function analytic and bounded in* Γ *and equal to $f(z)$ in G.*

The proof of this theorem is much the same as for Theorem A taken together with the remark of §5 in the earlier paper and is therefore omitted.

The conclusion of Theorem D simply means of course that $f(z)$ shall be analytically extensible throughout Γ.

UNION COLLEGE

A GENERALIZED PARSEVAL'S RELATION

BY E. S. QUADE

A function $\phi(x)$ which is non-negative, convex, and satisfies the conditions $\phi(0) = 0$ and $(\phi(x)/x) \to \infty$ as $x \to \infty$ will be called a Young's function. Given such a function $\phi(x)$, a second function, $\psi(x)$, with the same properties can be found such that Young's inequality, $ab \leqq \phi(a) + \psi(b)$, holds for every $a, b \geqq 0$. The functions $\phi(x)$ and $\psi(x)$ are then said to be complementary in the sense of Young.†

If $x(t)$ is such that $\int_a^b \phi(|x|)dt$ exists, $x(t)$ is said to belong to the space $L_\phi(a, b)$. This space is not necessarily linear.‡ For this reason we denote by $L_\phi^*(a, b)$ the class of all functions $x(t)$, $a \leqq t \leqq b$, such that the product $x(t)y(t)$ is integrable for every $y(t) \epsilon L_\psi(a, b)$. If we put

$$\|x\|_\phi = \sup_y \left| \int_a^b x(t)y(t)dt \right|$$

for all $y(t)$ with

$$\rho_y \equiv \int_a^b \psi(|y|)dt \leqq 1,$$

then L_ϕ^* is a linear metric, and complete space.§ A function

† W. H. Young, Proceedings Royal Society, (A), vol. 87 (1912), pp. 225–229.

‡ W. Orlicz, *Über eine gewisse Klasse von Räumen vom Typus B*, Bulletin, Académie Polonaise, (A), Cracovie (1932).

§ A. Zygmund, *Trigonometrical Series*, 1935, pp. 95–100.

$x(t) \epsilon L_\phi^*$ can be approximated by its Féjer polynomials;† that is, given $\epsilon > 0$, there exists an $N_0(\epsilon)$ such that

$$\|x - \sigma_n\|_\phi \leqq \epsilon, \qquad n \geqq N_0(\epsilon),$$

where $\sigma_n = \sigma_n(t; x)$ is the nth $(C, 1)$ mean of the Fourier series of $x(t)$. Since $\sigma_n(t; x)$ is continuous, there exists a step function $\omega_n(t)$ which is a linear combination of simple step functions such that $|\sigma_n(t; x) - \omega_n(t)| \leqq \epsilon$ for $a \leqq t \leqq b$. Then

$$\begin{aligned}\|x - \omega_n\|_\phi &\leqq \|x - \sigma_n\|_\phi + \|\sigma_n - \omega_n\|_\phi \\ &\leqq \epsilon + \sup_y \left| \int_a^b [\sigma_n(t) - \omega_n(t)] y(t) dt \right| \\ &\leqq \epsilon + \epsilon \left(\sup_y \int_a^b |y(t)| \, dt \right) \\ &\leqq \epsilon + \epsilon[\phi(1)(b - a) + 1] = M\epsilon,\end{aligned}$$

where M is independent of $x(t)$. This implies that the set of simple step functions is a fundamental set in the space L_ϕ^*.

In the following we take (0, 1) as the fundamental interval and $\{K_n(s, t)\}$ to be a sequence of measurable kernels defined on the unit square.

THEOREM 1. *Necessary and sufficient conditions that*

$$\int_0^1 y(s) ds \int_0^1 K_n(s, t) x(t) dt \to \int_0^1 x(s) y(s) ds$$

for every pair $x(t) \epsilon L_\psi^*$, $y(s) \epsilon L_\phi^*$ are as follows.

(1) *If $\delta = (a, b)$ and $\pi = (\alpha, \beta)$ are any subintervals of (0, 1), then*

$$\int_a^b ds \int_\alpha^\beta K_n(s, t) dt \to \text{meas } (\delta \cdot \pi);$$

$$\left\| \int_0^1 K_n(s, t) x(t) dt \right\|_\psi \leqq M \|x\|_\psi, \tag{2}$$

where M is independent of n.‡

† A. Zygmund, loc. cit., p. 107 (14).

‡ For analogous results in some more special cases we refer to J. C. Burhill, *Strong and weak convergence of functions of general type*, Proceedings London Mathematical Society, (2), vol. 28 (1928), pp. 493–500; Z. W. Birnbaum and W. Orlicz, *Über die Verallgemeinerung des Begriffes der zueinander konjugierten Potenzen*, Studia Mathematica, vol. 3 (1931), pp. 1–67; W. Orlicz, loc. cit.

Since the set of simple step functions is a fundamental set, the necessity of (1) is obvious. In order that the integral

$$\int_0^1 y(s)ds \int_0^1 K_n(s, t)x(t)dt$$

exist for arbitrary $y(s) \epsilon L_\psi^*$,

$$U_n(x) \equiv \int_0^1 K_n(s, t)x(t)dt$$

must belong to L_ψ^* for arbitrary $x(t) \epsilon L_\psi^*$ and n. We wish to show that $U_n(x)$ is a linear operation on L_ψ^* to L_ψ^*. We need the following lemmas.

LEMMA A. $\lim_{n\to\infty} \|x_n\|_\psi = 0$ *implies* $\lim \operatorname{asymp}_{n\to\infty} x_n(t) = 0$.

Set $y_n(t) = a \operatorname{sign} x_n(t)$, where $a > 0, \psi(a) \leqq 1$. Then

$$\|x_n\|_\psi = \sup_y \left| \int_0^1 y(t)x_n(t)dt \right| \geqq \left| \int_0^1 y_n(t)x_n(t)dt \right|$$

$$= a \int_0^1 |x_n(t)| \, dt \to 0.$$

But

$$\lim_{n\to\infty} \int_0^1 |x_n(t)| \, dt = 0 \text{ implies } \lim \operatorname{asymp}_{n\to\infty} x_n(t) = 0.$$

LEMMA B. $\lim_{n\to\infty} \|x_n\|_\psi = 0$ *implies the existence in* L_ψ^* *of a subsequence* $\{x_{n_i}(t)\}$ *and an* $x(t)$ *such that* $|x_{n_i}(t)| \leqq |x(t)|$ *for every* $i = 1, 2, 3, \cdots$, *and almost all* t.

Let $\|x_n(t)\|_\psi \to 0$. Without loss of generality we may take each $x_n(t) \geqq 0$. Then there exists a subsequence $\{x_{n_i}(t)\} \subset \{x_n(t)\}$ such that $\sum_{i=1}^\infty \|x_{n_i}\|_\psi \leqq M$, where M is a constant. This means that $\sum_{i=1}^\infty x_{n_i}(t)$ converges in the space L_ψ^* to a function $x(t) \epsilon L_\psi^*$. Then

$$|x_{n_i}(t)| = x_{n_i}(t) \leqq x(t).$$

LEMMA C. $\lim \operatorname{asymp}_{n\to\infty} x_n(t) = x(t)$ *implies* $\underline{\lim}_{n\to\infty} \|x_n\|_\psi \geqq \|x\|_\psi$.

Suppose the contrary. Then there exists a subsequence $\{x_{n_i}(t)\}$ such that $\lim_{i\to\infty}\|x_{n_i}\|_\psi < \|x\|_\psi$. This means that

$$\sup_y \left| \int_0^1 x_{n_i} y \, dt \right| \leqq \|x\|_\psi \text{ which implies } \int_0^1 |x_{n_i} y| \, dt \leqq \|x\|_\psi$$

for all i sufficiently large. Since $\lim \operatorname{asymp}_{i\to\infty} x_{n_i}(t) = x(t)$, there exists a subsequence $x_{n_i}(t) \to x(t)$ almost everywhere. Hence $|x_{n_i}(t)y(t)| \to |x(t)y(t)|$ almost everywhere for each fixed $y(t)$. By the Fatou Lemma we have

$$\underline{\lim}_{i\to\infty} \int_0^1 |x'_{n_i} y| \, dt \geqq \int_0^1 |xy| \, dt \geqq \left| \int_0^1 xy dt \right|.$$

Since this must hold for every $y(t)$, we have

$$\lim_{i\to\infty} \|x'_{n_i}\|_\psi \geqq \sup_y \lim_{i\to\infty} \int_0^1 |x'_{n_i} y| \, dt \geqq \|x\|_\psi,$$

which means that $\lim_{i\to\infty}\|x_{n_i}\|_\psi \geqq \|x\|_\psi$, a contradiction.

Consequently, by a general theorem of Banach,† $U_n(x)$ is a linear operation. Hence $\|U_n(x)\|_\psi \leqq M_n\|x\|_\psi$. But the sequence

$$\left\{ \left| \int_0^1 y(s)dt \int_0^1 K_n(s, t)x(t)dt \right| \right\}$$

is bounded (for each fixed $x(t) \in L_\psi^*$) for every $y(s) \in L_\phi^*$. This implies‡ $\|U_n(x)\|_\psi \leqq M(x)$ independently of n. Then, by the Banach-Steinhaus theorem,

$$\|U_n(x)\|_\psi = \left\| \int_0^1 K_n(s, t)x(t)dt \right\|_\psi \leqq M\|x\|_\psi,$$

where M is independent of n.

To prove the sufficiency we set

$$\int_\alpha^\beta K_n(s, t)dt = \int_0^1 K_n(s, t)w_{\alpha\beta}(t)dt = w_n(s),$$

† S. Banach, *Théorie des Opérations Linéaires*, 1932, p. 87.

‡ A. Zygmund, loc. cit., p. 99.

where $w_{\alpha\beta}(t)$ is the characteristic function of the interval (α, β). From (1) we have

$$\int_a^b w_n(s)ds \to \int_a^b w_{\alpha\beta}(s)ds.$$

Let

$$f_n(y) = \int_0^1 y(s)ds \int_\alpha^\beta K_n(s, t)dt = \int_0^1 y(s)w_n(s)ds.$$

Then if $\lambda \equiv M\|w_{\alpha\beta}\|_\psi$ and $\rho'_{w_n/\lambda} = \max(1, \rho'_{w_n/\lambda})$, since

$$\rho_{w_n/\lambda} \equiv \int_0^1 \psi\left(\frac{1}{\lambda}|\, w_n(t)\,|\right)dt \leqq 1,$$

we have†

$$|\, f_n(y)\,| \leqq \lambda\|y\|_\phi \rho'_{w_n/\lambda} \leqq M\|w_{\alpha\beta}\|_\psi \|y\|_\phi,$$

where the right-hand side of the inequality is independent of n. Hence from a general theorem on linear functionals,‡ we have

$$\int_0^1 y(s)w_n(s)ds \to \int_0^1 y(s)w_{\alpha\beta}(s)ds$$

for arbitrary $y(s) \in L_\phi^*$. Now set

$$g_n(x) = \int_0^1 y(s)ds \int_0^1 K_n(s, t)x(t)dt.$$

This gives a sequence of linear functionals defined on L_ψ^*. For every $w_{\alpha\beta}(s)$,

$$g_n(w_{\alpha\beta}) \to g(w_{\alpha\beta}) \equiv \int_0^1 y(s)w_{\alpha\beta}(s)ds.$$

Moreover, since§

$$|\, g_n(x)\,| \leqq \rho'_y \left\| \int_0^1 K_n(s, t)x(t)dt \right\|_\psi \leqq \rho'_y M\|x\|_\psi,$$

† A. Zygmund, loc. cit., p. 97.
‡ S. Banach, loc. cit., p. 123, Theorem 2.
§ A. Zygmund, loc. cit., p. 97.

we see that the modulus of the functional $g_n(x)$ is $\leqq M_{\rho_y}'$ independently of n, and we may again apply the theorem on functionals used above to obtain the desired result.

As a corollary of Theorem 1, we have the following theorem.

THEOREM 2. *Necessary and sufficient conditions that*

$$\int_0^1 y(s)ds \int_0^1 K_n(s, t)x(t)dt \to \int_0^1 y(s)x(s)ds$$

for every pair $y(s) \epsilon L_p$, $x(s) \epsilon L_{p'}$, $(1 < p < \infty)$, *are*:

(1) $$\int_a^b ds \int_\alpha^\beta K_n(s, t)dt \to \text{meas } (\delta \cdot \pi);$$

(2) $$\left(\int_0^1 \left| \int_0^1 K_n(s, t)x(t)dt \right|^{p'} ds \right)^{1/p'} \leqq M \| x \|_{p'}.$$

[For the case $x(s) \epsilon L$, $y(s) \epsilon M \equiv L_\infty$, the theorem holds provided we replace in (1) the interval (a, b) by the arbitrary measurable set E. The proof for this case is essentially the same as that of Theorem 1.]

THEOREM 3. *Necessary and sufficient conditions*† *that*

$$\int_0^1 y(s)ds \int_0^1 K_n(s, t)x(t)dt \to \int_0^1 y(s)x(s)ds$$

for every pair $y(s) \epsilon L$, $x(t) \epsilon M$ *are*:

(1) $$\int_a^b ds \int_E K_n(s, t)dt \to \text{meas } \delta \cdot E;$$

(2) $$\operatorname{ess\,sup}_s \int_0^1 | K_n(s, t) | \, dt \leqq M.$$

Except for the necessity of (2) the proof is exactly that of Theorem 1. As in Theorem 1 we have

$$\operatorname{ess\,sup}_s \left| \int_0^1 K_n(s, t)x(t)dt \right| \leqq M' \| x \|_M.$$

† J. D. Tamarkin, Zentralblatt, vol. 10 (1935), pp. 349–350, in review of paper by I. Natanson, Bulletin, Société de Physique et Mathématique, Kazan, vol. 3 (1934).

Letting $x(t)=e(t)$ be the characteristic function for the set E, we have

$$\operatorname*{ess\,sup}_{s}\left|\int_0^1 K_n(s,t)dt\right| \leqq M'.$$

This implies† that (2) holds.

If, in Theorem 2, condition (2) were replaced by either

$$(2') \qquad \left(\int_0^1\left(\int_0^1 |K_n(s,t)|^p dt\right)^{p'/p} ds\right)^{1/p'} \leqq M, \qquad (n=1,2,\cdots),$$

or

$$(2'') \qquad \left(\int_0^1\left(\int_0^1 |K_n(s,t)|^{p'} ds\right)^{p/p'} dt\right)^{1/p} \leqq M, \qquad (n=1,2,\cdots),$$

the conditions of Theorem 2 would be sufficient but no longer necessary. For example, the Dirichlet kernel, $D_n(s,t)=D_n(t,-s)$, (where the fundamental interval is $(0, 2\pi)$ and the functions are assumed periodic) does not satisfy either of these conditions but does satisfy those of Theorem 2. We note that for $p\geqq 2$, $(2'')$ implies $(2')$ and for $1<p\leqq 2$, $(2')$ implies $(2'')$.

THEOREM 4. *Sufficient conditions that*

$$\int_0^1 y(s)ds\int_0^1 K_n(s,t)x(t)dt \to \int_0^1 y(s)x(s)ds,$$

where $x(s)\in L_\psi^*$, $y(s)\in L_\phi^*$ *are*:

$$(1) \qquad \int_a^b ds\int_\alpha^\beta K_n(s,t)dt \to \text{meas } (\delta\cdot\pi);$$

$$(2) \qquad \operatorname*{ess\,sup}_{s}\int_0^1 |K_n(s,t)|\,dt \leqq M, \qquad (n=1,2,\cdots);$$

$$(3) \qquad \operatorname*{ess\,sup}_{t}\int_0^1 |K_n(s,t)|\,ds \leqq M, \qquad (n=1,2,\cdots).$$

† Saks and Tamarkin, Annals of Mathematics, vol. 34 (1933), p. 600. Theorem 2.

We set $u_n(s) = \int_0^1 K_n(s, t)x(t)dt$ and consider

$$\frac{u_n(s)}{M} = \int_0^1 \left(\frac{K_n(s, t)}{M}\right) x(t)dt.$$

Since

$$\operatorname*{ess\,sup}_s \int_0^1 | K_n(s, t) | \, dt \leqq 1$$

and $\psi(0) = 0$, we have by Jensen's inequality

$$\begin{aligned}
\psi\left(\frac{|u_n(s)|}{M\|x\|_\psi}\right) &= \psi\left(\left|\int_0^1 \frac{K_n(s, t)}{M} \cdot \frac{x(t)}{\|x\|_\psi} dt\right|\right) \\
&\leqq \psi\left(\int_0^1 \left|\frac{K_n(s, t)}{M}\right| \cdot \left|\frac{x(t)}{\|x\|_\psi}\right| dt\right) \\
&\leqq \frac{1}{M}\int_0^1 \psi\left(\frac{|x(t)|}{\|x\|_\psi}\right) | K_n(s, t) | \, dt.
\end{aligned}$$

Consequently

$$\begin{aligned}
\int_0^1 \psi\left(\frac{|u_n(s)|}{M\|x\|_\psi}\right) ds &\leqq \frac{1}{M}\int_0^1 ds \int_0^1 \psi\left(\frac{|x(t)|}{\|x\|_\psi}\right) | K_n(s, t) | \, dt \\
&\leqq \frac{1}{M}\int_0^1 \psi\left(\frac{|x(t)|}{\|x\|_\psi}\right) dt \int_0^1 | K_n(s, t) | \, ds \\
&\leqq \int_0^1 \psi\left(\frac{|x(t)|}{\|x\|_\psi}\right) dt \leqq 1.
\end{aligned}$$

We have thus shown that $u_n(s) = \int_0^1 K_n(s, t)x(t)dt$ takes $x(t) \in L_\psi^*$ into $u_n(s) \in L_\psi^*$ for arbitrary n, and hence condition (2) of Theorem 1 follows as in the proof of the necessity in that theorem.

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ON THE COMPOSITION OF QUADRATIC FORMS*

BY E. D. JENKINS

1. *Introduction.* A compound of two binary quadratic forms $f_1(x_1, y_1)$ and $f_2(x_2, y_2)$ is a third form $f(x, y)$ such that $f(x, y) = f_1(x_1, y_1)f_2(x_2, y_2)$ under a primitive bilinear transformation

$$x = p_1x_1x_2 + p_2x_1y_2 + p_3y_1x_2 + p_4y_1y_2,$$
$$y = q_1x_1x_2 + q_2x_1y_2 + q_3y_1x_2 + q_4y_1y_2.$$

The fundamental problem of this theory is to determine a compound of two given forms and the transformation under which the relationship exists.

This problem has been considered† by Gauss, Arndt, Dedekind and others. The method of Dedekind‡ was based upon a correspondence between forms and moduls in an algebraic field, composition of forms corresponding to multiplication of moduls.

The method of the present paper is also based upon a correspondence between forms and moduls. These moduls form a subclass of those considered by Dedekind in that only integral moduls are employed. These moduls in turn are in correspondence with matrices with rational integral elements, the greatest common divisor process corresponding to multiplication of moduls and hence to composition of forms. By this correspondence and the matric g. c. d. process of Châtelet,§ the composition of quadratic forms requires but a small fraction of the time and labor required by previously known methods.

2. *The Correspondence.* Let $1, \theta$ be an integral basis for the quadratic field $\mathfrak{F}(\sqrt{m})$. Then $\theta^2 = m$ if $m \equiv 2, 3 \pmod 4$, (Case 1); and $\theta^2 = \theta + m', 4m' = m - 1$ if $m \equiv 1 \pmod 4$, (Case 2). Let a, b, and k be rational integers, of which a and k are positive, such that $b^2 - k^2 \equiv 0 \pmod a$, (Case 1); $b^2 + bk - k^2m' \equiv 0 \pmod a$, (Case 2).

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* Presented to the Society, April 19, 1935.

† Gauss, *Disquisitiones Arithmeticae*, 1801, §234 ff.; Arndt, *Auflösung einer Aufgabe in der Composition der quadratischen Formen*, Journal für Mathematik, vol. 56 (1859), pp. 64–71.

‡ Dirichlet, *Vorlesungen über Zahlentheorie*, 4th ed., 1894, pp. 640–657.

§ A. Châtelet, *Groupes Abéliens Finis*, 1924, p. 26.

The general number $\alpha = \alpha_1 x + \alpha_2 y$ of the modul with basis

$$\alpha_1 = a, \qquad \alpha_2 = b + k\theta \tag{1}$$

has the norm

$$N(\alpha) = a\left(ax^2 + 2bxy + \frac{b^2 - k^2 m}{a} y^2\right),$$

$$N(\alpha) = a\left(ax^2 + (2b + k)xy + \frac{b^2 + bk - k^2 m'}{a} y^2\right),$$

in Cases 1 and 2, respectively. We shall say that the form

$$\begin{aligned} f(x, y) &= ax^2 + 2bxy + \frac{b^2 - k^2 m}{a} y^2, \\ f(x, y) &= ax^2 + (2b + k)xy + \frac{b^2 + bk - k^2 m'}{a} y^2, \end{aligned} \tag{2}$$

is associated with the basis (1) and also with the matrix

$$G = \begin{pmatrix} a & 0 \\ b & k \end{pmatrix}. \tag{3}$$

The fact that we consider only irreducible forms makes it necessary that $a \neq 0$ and $k \neq 0$.

The totality of bases of the above modul is given by

$$\beta_1 = c_{11}\alpha_1 + c_{12}\alpha_2, \qquad \beta_2 = c_{21}\alpha_1 + c_{22}\alpha_2, \tag{4}$$

where α_1, α_2 is the basis (1) and the c_{ij} are rational integers of determinant ± 1. Then all bases of the modul have matrices of the form CG, where $C = (c_{ij})$ is a unimodular matrix with rational integral elements.

Let $\beta = \beta_1 x_1 + \beta_2 y_1$. Define the transformation

$$X = c_{11}x_1 + c_{21}y_1, \qquad Y = c_{12}x_1 + c_{22}y_2 \tag{5}$$

of determinant ± 1. Then $\beta \quad \alpha_1 X + \alpha_2 Y$, and in Cases 1 and 2, respectively,

$$\begin{aligned} f_1(x_1, y_1) = \frac{N(\beta)}{a} &= aX^2 + 2bXY + \frac{b^2 - k^2 m}{a} Y^2, \\ &= aX^2 + (2b + k)XY + \frac{b^2 + bk - k^2 m'}{a} Y^2, \end{aligned}$$

is a quadratic form which, by (5), is either properly or improperly equivalent to $f(x, y)$. Since every basis of the modul can be reduced by a transformation of determinant $+1$ either to the basis $(a, b+k\theta)$ or to the basis $(-a, b+k\theta)$, the form represented by any basis of the modul is properly equivalent either to the form (2) or to its opposite,

$$g(x, y) = ax^2 - 2bxy + \frac{b^2 - k^2m}{a} y^2,$$

$$= ax^2 - (2b + k)xy + \frac{b^2 + bk - k^2m'}{a} y^2,$$

in Cases 1 and 2, respectively. However, the form $g(x, y)$ can be represented by the matrix

$$G' = \begin{pmatrix} a & 0 \\ -b & k \end{pmatrix}, \text{(Case 1)}, \quad G' = \begin{pmatrix} a & 0 \\ -(b+k) & k \end{pmatrix}, \text{(Case 2)},$$

with a and k positive, so we need employ only matrices of type (3), or matrices which can be reduced to type (3) by a transformation of determinant $+1$.

Next, consider an irreducible form

$$f(x, y) = ax^2 + Bxy + cy^2, \qquad (a > 0). \tag{6}$$

The problem is to determine what integral algebraic modul will define the above form.

Consider the discriminant $d = B^2 - 4ac = l^2m$, where m contains no square factor greater than 1. Consequently $m \equiv 1, 2, \text{or } 3 \pmod 4$. If $m \equiv 1 \pmod 4$, $B^2 \equiv l^2 \pmod 4$ implies $B - l$ is even. Hence we define integers b and k such that $k = l$, $2b + k = B$. If $m \equiv 2, 3 \pmod 4$, $B^2 \equiv l^2m \pmod 4$ can hold only if B and l are both even. Hence we define $2k = l$, $2b = B$. In either case

$$\alpha_1 = a, \qquad \alpha_2 = b + k\theta$$

is a modul of type (1), whose existence we supposed, and this modul yields form (6).

We wish to determine a necessary and sufficient condition that a matrix of positive determinant and having k as the positive g. c. d. of the elements of the second column, correspond to a quadratic form.

Let Γ be the matrix*

* C. C. MacDuffee, Annals of Mathematics, (2), vol. 29 (1928), p. 200.

$$\Gamma = \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ m' & 1 \end{pmatrix}$$

for Cases 1 and 2. Consider first a matrix of type (3). In Case 1, let

$$\begin{pmatrix} a & 0 \\ b & k \end{pmatrix}\begin{pmatrix} 0 & k \\ km & 0 \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a & 0 \\ b & k \end{pmatrix}.$$

Then

$$0 = ea + fb, \qquad ak = fk,$$
$$k^2m = ga + hb, \qquad bk = hk.$$

Since $k \neq 0$, we see that $f = a$, $h = b$, $e = -b$, $g = -(b^2 - k^2m)/a$ are rational integers if G corresponds to a form. Conversely if

$$\begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

is a matrix with rational integral elements, $b^2 - k^2m \equiv 0 \pmod{a}$ and G is a matrix of a form. The proof for Case 2 is similar.

If G' is any non-singular matrix of positive determinant, then* $G' = CG$, where C is an integral matrix of determinant $+1$, and G is a matrix of type (3). If G' is a matrix of a form, then $G = G^1G'$ is also a matrix of a form, whence, by the above proof,

$$Gk\Gamma = \Lambda' G, \tag{7}$$

where Λ' is a matrix with rational integral elements. Hence

$$G'k\Gamma = C\Lambda' C^1 G' \tag{8}$$

where $C\Lambda C^1$ is also a matrix with integral elements. Since (8) also implies (7), we have proved the following theorem.

THEOREM 1. *A necessary and sufficient condition that a non-singular matrix G', of positive determinant, the g. c. d. of the elements of whose second column is k, represent a quadratic form of discriminant $4k^2m$ (Case 1), k^2m (Case 2), is that*

$$G'k\Gamma = \Lambda G',$$

where Λ is a matrix with rational integral elements.

* C. C. MacDuffee, *Theory of Matrices*, 1933, p. 32.

3. *Product of Moduls.* We define the second or S-matrix of any algebraic number $\beta = b_1 + b_2\theta$ to be the matrix

$$S(\beta) = b_1\mathrm{I} + b_2\Gamma,$$

where I is the identity matrix and Γ is defined as above. The theorem which was proved* for ideal multiplication in a linear algebra holds for the multiplication of integral algebraic moduls. That is, if G_1 and G_2 are the basis matrices of the moduls $M_1 = [\alpha_1, \alpha_2]$ and $M_2 = [\beta_1, \beta_2]$, a matrix of the product modul M is a greatest common right divisor of $G_1S(\beta_1)$ and $G_1S(\beta_2)$. We shall prove the following theorem.

THEOREM 2. *A matric g. c. r. d. (of positive determinant) of $G_1S(\beta_1)$ and $G_1S(\beta_2)$ represents a quadratic form.*

Let the two given matrices, representing forms $f_1(x_1, y_1)$ and $f_2(x_2, y_2)$ be

$$G_1 = \begin{pmatrix} a_1 & 0 \\ b_1 & k_1 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} a_2 & 0 \\ b_2 & k_2 \end{pmatrix}.$$

The matric g. c. r. d. of $G_1S(\beta_1)$ and $G_1S(\beta_2)$ may be put into the form

$$G = \begin{pmatrix} a & 0 \\ b & k \end{pmatrix},$$

where a and k are positive integers. Moreover†

$$\begin{pmatrix} G_1S(\beta_1) & 0 \\ G_1S(\beta_2) & 0 \end{pmatrix} \underset{\mathrm{L}}{=} \begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix}. \tag{9}$$

Case 1.

$$G_1S(\beta_1) = \begin{pmatrix} a_1a_2 & 0 \\ b_1a_2 & k_1a_2 \end{pmatrix}, \quad G_1S(\beta_2) = \begin{pmatrix} a_1b_2 & a_1k_2 \\ b_1b_2 + k_1k_2m & b_1k_2 + k_1b_2 \end{pmatrix}.$$

By (9), comparing elements in the first row and second column on both sides of the matric equation, we see that $0 = q_1k$. Since $k \neq 0$, $q_1 = 0$, and by comparing elements in the first row and first column, we see that there exists an integer p_1 such that

* Grace Shover and C. C. MacDuffee, this Bulletin, vol. 37 (1931), pp. 434–38.

† A. Châtelet, *Groupes Abéliens Finis*, 1924, p. 26.

$$a_1a_2 = p_1a. \tag{10}$$

Also from (9) it is seen that there exist rational integers p, q, r, s such that

$$\begin{aligned} b &= pa_1a_2 + qb_1a_2 + ra_1b_2 + s(b_1b_2 + k_1k_2m), \\ k &= qk_1a_2 + ra_1k_2 + s(b_1k_2 + k_1b_2). \\ b^2 - k^2m &\equiv q^2a_2^2(b_1^2 - k_1^2m) + r^2a_1^2(b_2^2 - k_2^2m) \\ &\quad + s^2(b_1^2 - k_1^2m)(b_2^2 - k_2^2m) \\ &\quad + 2qsa_2b_2(b_1^2 - k_1^2m) + 2rsa_1b_1(b_2^2 - k_2^2m) \\ &\equiv q^2a_2^2a_1c_1 + r^2a_1^2a_2c_2 + s^2a_1a_2c_1c_2 + 2qsb_2c_1a_1a_2 \\ &\quad + 2rsb_1c_2a_1a_2 \\ &\equiv 0 \pmod{a_1a_2} \equiv 0 \pmod{a}. \end{aligned}$$

Similarly, in Case 2, $b^2+bk-k^2m' \equiv 0 \pmod{a}$.

Since the g. c. r. d., taken in the form above, represents a quadratic form, any g. c. r. d. of positive determinant will also represent a form.

Let the general numbers of the moduls represented by matrices G_1, G_2, and G be

$$\alpha = a_1x_1 + (b_1 + k_1\theta)y_1, \qquad \beta = a_2x_2 + (b_2 + k_2\theta)y_2,$$
$$\gamma = ax + (b + k\theta)y.$$

There exists a primitive bilinear transformation under which $\alpha\beta=\gamma$. Then, since $N(\alpha)N(\beta)=N(\gamma)$, it follows that

$$a_1a_2f_1(x_1, y_1)f_2(x_2, y_2) = a\phi(x, y),$$

where ϕ is the form represented by matrix G. By (10), we have the following theorem.

THEOREM 3. *The form $\phi(x, y)$, represented by a matric g. c. r. d. of $G_1S(\beta_1)$ and $G_1S(\beta_2)$, is such that*

$$p_1f_1(x_1, y_1)f_2(x_2, y_2) = \phi(x, y),$$

where p_1 is the integer defined by (10).

Dedekind has established* a correspondence between binary

* Dirichlet, *Zahlentheorie*, 4th ed., §187.

quadratic forms and algebraic moduls. The moduls we consider form a subset of those considered by Dedekind, but a subset which is very convenient from the computational standpoint.

Let the matrix G, given by (3), correspond to the form $f(x, y)$. By the Dedekind theory, the algebraic number $\omega = (b/a) + (k/a)\theta$ satisfies the irreducible equation

$$\frac{1}{l}\left(a\omega^2 - 2b\omega + \frac{b^2 - k^2m}{a}\right) = 0, \qquad \text{(Case 1)},$$

$$\frac{1}{l}\left(a\omega^2 - (2b + k)\omega + \frac{b^2 + bk - k^2m'}{a}\right) = 0, \qquad \text{(Case 2)},$$

where l is the g. c. d. of a, $2b$, $(b^2 - k^2m)/a$, or of a, $2b+k$, $(b^2+bk-k^2m')/a$ for the respective cases. If $f'(x, y)$ denotes the form corresponding to the matrix G under the Dedekind theory, $f(x, y) = lf'(x, y)$, where l is the g. c. d. of the coefficients of $f(x, y)$. It is also clear* that for the modul M, corresponding to G, the norm $N(M) = a^2 \div (a/l) = al$. Since

$$N(M_1)N(M_2) = N(M),$$

where modul M is the product of moduls M_1 and M_2, it follows that $a_1a_2l_1l_2 = al$. Since, by (10), $a_1a_2 = p_1a$, the above equation becomes

$$p_1l_1l_2 = l \quad \text{or} \quad l_1l_2 = l/p_1. \tag{11}$$

If f_1', f_2', and f' are the forms corresponding to moduls M_1, M_2, and M under the Dedekind theory, then $f_1'f_2' = f'$, where f' is a compound of f_1' and f_2'. Multiplying both sides of this equation by (11_2), we have $f_1(x_1, y_1)f_2(x_2, y_2) = f(x, y)$, where f is a compound of f_1 and f_2. But f_1 and f_2 correspond to matrices G_1 and G_2, while the form $\phi(x, y) = lf' = p_1f(x, y)$ corresponds to G. Hence we have the following theorem.

THEOREM 4. *The form $\phi(x, y)$ which corresponds to a g. c. r. d. of $G_1S(\beta_1)$ and $G_1S(\beta_2)$ is such that $\phi(x, y) = p_1f(x, y)$, where $f(x, y)$ is a compound of $f_1(x_1, y_1)$ and $f_2(x_2, y_2)$, and $p_1 = a_1a_2/a$ is an integer.*

4. *Conclusion.* A computational method for the composition of forms is now at our disposal. Furthermore, if we restrict our

* Ibid.

matrices to reduced forms, the element in the upper right corner being a 0, the form may be read directly from the matrix without reference to the modul it defines. For other forms of the matrix we can find the corresponding binary form from the norm of the general number defined by the matrix. In either case we have a simple and direct method for determining a compound of two given forms.

As an example, let us take $f_1(x_1, y_1) = 5x_1^2 + 2x_1y_1 + 5y_1^2$, and $f_2(x_2, y_2) = 77x_2^2 + 20x_2y_2 + 2y_2^2$. Their discriminants are $d_1 = -96 = 16(-6)$ and $d_2 = -216 = 36(-6)$, so the forms are associated with the field $\mathfrak{F}(\sqrt{-6})$. Since $-6 \equiv 2 \pmod 4$,

$$\Gamma = \begin{pmatrix} 0 & 1 \\ -6 & 0 \end{pmatrix}.$$

We note that $16(-6) = 4k_1^2 m = -24k_1^2$ and $36(-6) = -24k_2^2$, whence $k_1 = 2$, $k_2 = 3$. Then

$$G_1 = \begin{pmatrix} 5 & 0 \\ 1 & 2 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} 77 & 0 \\ 10 & 3 \end{pmatrix}.$$

$$G_1 S(\beta_1) = \begin{pmatrix} 385 & 0 \\ 77 & 154 \end{pmatrix}, \qquad G_1 S(\beta_2) = \begin{pmatrix} 50 & 15 \\ -26 & 23 \end{pmatrix}.$$

$$\begin{pmatrix} 385 & 0 & 0 & 0 \\ 77 & 154 & 0 & 0 \\ 50 & 15 & 0 & 0 \\ -26 & 23 & 0 & 0 \end{pmatrix} \underset{=}{L} \begin{pmatrix} 385 & 0 & 0 & 0 \\ 183 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so

$$G = \begin{pmatrix} 385 & 0 \\ 183 & 1 \end{pmatrix},$$

and, since $p_1 = 1$, the compound is the form

$$f = 385x^2 + 366xy + 87y^2.$$

By equating the coefficients of 1 and θ in the equation $\alpha\beta = \gamma$, the bilinear transformation is found to be

$$x = x_1x_2 - 7x_1y_2 - 73x_2y_1 - 11y_1y_2,$$
$$y = \qquad 15x_1y_2 + 154x_2y_1 + 23y_1y_2.$$

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ON CONTINUED FRACTIONS OF THE FORM

$$1 + \mathop{K}_{1}^{\infty} (b_n z/1)$$

BY H. S. WALL

1. *Introduction.* The principal object of this paper is to determine the region of convergence of the infinite continued fraction

$$(1) \qquad 1 + \mathop{K}_{1}^{\infty} (b_n z/1) = 1 + \frac{b_1 z}{1} + \frac{b_2 z}{1} + \cdots, \qquad (b_n \neq 0),$$

when $b_1, b_2, b_3, \cdots$ are real or complex numbers such that for some $k \geqq 1$

$$(2) \qquad \lim_{n=\infty} b_{nk+m} = \sigma_m, \qquad (m = 1, 2, 3, \cdots, k).$$

The results may be stated in terms of the numerators and denominators $u_{n,\lambda}$, $v_{n,\lambda}$ of the nth convergent of the continued fraction $1+K_{\nu=1}^{\infty}(\sigma_{\nu+\lambda}z/1)$, $(\sigma_{nk+m}=\sigma_m)$, as follows.

THEOREM 1. *Let*† *us write* $G_k = v_{k-1,1}v_{k-1,2} \cdots v_{k-1,k}$ *and* $H_k = v_k + u_{k-1} - v_{k-1}$; *and let us set*

$$Z(z) = -(-z)^k \sigma_1 \sigma_2 \cdots \sigma_k / H_k^2 .$$

Let R be an arbitrary bounded closed and connected region of the z plane containing the origin on the interior and which contains (within or upon the boundary) none of the zeros of the polynomials G_k, H_k, *nor points z such that* $Z(z)$ *is a real number* $\leqq -1/4$. *Then* (1) *converges over R except at certain isolated points* $p_1, p_2, \cdots, p_\mu$, *and uniformly over the region obtained from R by removing the interiors of small circles with centers* $p_1, p_2, \cdots, p_\mu$. *The limit is a non-rational function of z analytic over R except at* $p_1, p_2, \cdots, p_\mu$, *which are poles.*

The function $Z(z)$ determines a transformation of the z plane into the Z plane and $Z=Z(z)$. Except in the case $\sigma_1\sigma_2 \cdots \sigma_k = 0$, the set of points in the z plane such that Z is real and

† We write $u_{n,0}=u_n$, and $v_{n,0}=v_n$.

$\leqq -1/4$ is a portion of a curve C_k to which corresponds under the transformation the real Z segment $(-\infty, -1/4)$. The curve C_k is a stelloid† or Holzmüller‡ hyperbola in the plane of $1/z$. In particular, C_1 is a straight line, and C_2 a circle in the plane of z.

In §2 we prove Theorem 1; §3 contains a discussion of the curves C_k; and §4 contains examples and a discussion of the power series which corresponds to (1).

2. *Proof of Theorem* 1. If $N_n(z)/D_n(z)$ is the nth convergent of (1), there are k continued fractions

$$K_m = Z_0^m + \mathop{K}_{1}^{\infty}(Z_\nu^m/1), \qquad (m = 1, 2, \cdots, k),$$

with convergents $N_{\nu k+m-1}/D_{\nu k+m-1}$, $(\nu = 0, 1, 2, 3, \cdots)$, and §

$$Z_0^m = \frac{N_{m-1,0}}{D_{m-1,0}},$$

$$(3)\quad Z_1^m = \frac{(-1)^{m-1}b_1b_2\cdots b_m z^m D_{k-1,m}}{D_{m-1,0}D_{k+m-1,0}},$$

$$Z_n^m = \frac{-(-z)^k b_1^* b_2^* \cdots b_k^* D_{k-1,(n-3)k+m}D_{k-1,(n-1)k+m}}{D_{2k-1,(n-3)k+m}D_{2k-1,(n-2)k+m}},$$

for $n \geqq 3$, and where $b_j^* = b_{(n-2)k+m+j}$ and $N_{n,\lambda}/D_{n,\lambda}$ is the nth convergent of

$$(4)\qquad 1 + \frac{b_{1+\lambda}z}{1} + \frac{b_{2+\lambda}z}{1} + \cdots.$$

By a known theorem,‖ if

$$(5)\qquad \lim_{n=\infty} Z_n^m = Z(z)$$

uniformly over a region R', where $Z(z)$ is a continuous function having nowhere in R' a real value $\leqq -1/4$, then there exists an index N such that if $n \geqq N$, the continued fraction $K_{\nu=1}^{\infty}(Z_{\nu+n}^m/1)$ converges uniformly over R' to an analytic limit $F_n(z)$. Then if

† Fouret, Comptes Rendus, vol. 106 (1888).

‡ Holzmüller, *Einführung in die Theorie der isogonalen Verwandschaften*, 1882, p. 170 and p. 203.

§ Perron, *Die Lehre von den Kettenbruchen*, 1913, p. 200.

‖ Ibid., p. 285.

N_n'/D_n' is the nth convergent of K_m, the latter will converge over R' to the limit

$$\frac{N_n' + F_n N_{n-1}'}{D_n' + F_n D_{n-1}'}, \tag{6}$$

provided the denominator in (6) is not $\equiv 0$. But if, as we now suppose, R' contains the origin on the interior, this is impossible because the denominator $=1$ when $z=0$. Hence K_m converges to a function which is analytic except for poles, and clearly converges uniformly in the region obtained from R' by removing the interiors of small circles having these poles as centers. Also, K_m converges uniformly in the vicinity of the origin. From this it follows† that if R' is connected, and if (2) holds for $m=1, 2, 3, \cdots, k$ uniformly over R', and $Z(z)$ is independent of m, then $K_1 \equiv K_2 \equiv \cdots \equiv K_k \equiv P(z)$, where $P(z)$ is the power series corresponding to (1), and hence (1) converges after the manner of K_m to the same value.

It remains to be shown that under the hypothesis (2), (5) holds over the region R described in the theorem, and that $Z(z) = -(-z)^k \sigma_1 \sigma_2 \cdots \sigma_k / H_k^2$. We have, if $\delta_n = nk+m$,

$$D_{2k-1,\delta_n} = D_{k-1,\delta_{n+k}} D_{k,\delta_n} + (N_{k-1,\delta_{n+k}} - D_{k-1,\delta_{n+k}}) D_{k-1,\delta_n}, \tag{7}$$

and hence, if $D_{k-1,\delta_{n+1}}/D_{k-1,\delta_n} = 1+\epsilon_n = 1/(1+\epsilon_n')$,

$$Z_n^m = \frac{-(-z)^k b_1^* b_2^* \cdots b_k^*}{\Delta_{n-3}\ \Delta_{n-2}'},$$

where

$$\Delta_{n-3} = D_{k,\delta_{n-2}} + N_{k-1,\delta_{n-2}} - D_{k-1,\delta_{n-2}} + \epsilon_{n-3} D_{k,\delta_{n-2}},$$

$$\Delta_{n-2}' = D_{k,\delta_{n-2}} + N_{k-1,\delta_{n-1}} - D_{k-1,\delta_{n-1}} + \epsilon_{n-2}'(N_{k-1,\delta_{n-1}} - D_{k-1,\delta_{n-1}}).$$

By (2), $\lim_{n=\infty} \epsilon_n = \lim_{n=\infty} \epsilon_n' = 0$, $\lim_{n=\infty} N_{k,\delta_n} = u_{k,m}$, $\lim_{n=\infty} D_{k,\delta_n} = v_{k,m}$ uniformly over R. Also

$$v_{k,m} + u_{k-1,m} - v_{k-1,m} \equiv v_k + u_{k-1} - v_{k-1} \equiv H_k,$$

for all $m \geqq 1$. It now follows that over the region R

$$\lim_{n=\infty} Z_n^m = -(-z)^k \sigma_1 \sigma_2 \cdots \sigma_k / H_m^2, \qquad (m = 1, 2, \cdots, k),$$

uniformly, as was to be proved.

† Ibid., p. 342. The argument used there applies with slight modification to K_m.

In case $\sigma_1\sigma_2 \cdots \sigma_k = 0$, it is clear that $Z(z) \equiv 0$, and therefore can never be real and $\leqq -1/4$. In this case (5) holds uniformly over every bounded region from which the neighborhoods of the zeros of G_k, H_k have been excluded. It may happen that these neighborhoods need not be excluded. More generally, even if (2) fails to hold, we have the following theorem.

THEOREM 2. *If for some integer $k \geqq 1$ the functions Z_n^m, $(m = 1, 2, \cdots, k)$, defined in* (3) *converge uniformly to* 0 *for $n = \infty$ over every bounded region, then the continued fraction* (1) *represents a meromorphic function of z and converges except at the poles of that function.*

When $k = 1$ this reduces to the condition† $\lim b_n = 0$ found by E. B. Van Vleck. In §4 we shall give examples illustrating this theorem in the cases $k = 3, 4$.

3. *Discussion of the Curves C_k.* Put $p = (-1)^k\sigma_1\sigma_2 \cdots \sigma_k$, and let $p \neq 0$. By C_k' we shall understand the set of points in the z plane which is the image of the real Z segment $(-1/4, -\infty)$ under the transformation

$$Z = -pz^k/H_k^2 .$$

Then C_k' is a portion of a curve C_k which is the image of the negative half of the real Z axis under this transformation, and is a cut for the function represented by (1).

If (2) holds, then

$$\lim_{n=\infty} b_{nq+m} = \sigma_m', \qquad (m = 1, 2, 3, \cdots, q), \tag{8}$$

where $q = 2k$ and $\sigma_{m+k}' = \sigma_m' = \sigma_m$. If we had started with the hypothesis (8), then instead of the function $Z(z)$ we would have $Z'(z) = -p^2 z^q/H_q^2$; and

$$G_q = G_k^2 H_k^2, \tag{9}$$

$$H_q = H_k^2 - 2pz^k, \tag{10}$$

$$Z'(z) = -\frac{1}{(2 + 1/Z(z))^2}. \tag{11}$$

In fact, if we let $n = \infty$ in (7), we obtain the relation $v_{q-1,m} = v_{k-1,m}H_k$, from which (9) follows at once. From three relations analogous to (7) we obtain the identities

† Ibid., p. 345.

$$v_q = v_k^2 + v_{k-1}(u_k - v_k), \quad v_{q-1} = v_k v_{k-1} + v_{k-1}(u_{k-1} - v_{k-1}),$$
$$u_{q-1} = u_k v_{k-1} + u_{k-1}(u_{k-1} - v_{k-1}),$$

and consequently $H_q = H_k^2 + 2(u_k v_{k-1} - u_{k-1} v_k) = H_k^2 - 2pz^k$, which is (10). Finally, on eliminating $-pz^k$ between the relations

$$Z = -pz^k/H_k^2, \qquad Z' = -\left[-\frac{pz^k}{(H_k^2 - 2pz^k)}\right]^2,$$

we obtain (11).

It follows from (11) that C_k' is the same as C_q'; and from (9) we see that the zeros of G_q are those of $G_k H_k$. When $p \neq 0$, the zeros of H_k and of H_q lie on the cut; and when $p = 0$, the zeros of H_q are the same as those of H_k by (10). Hence when k is odd in (2) we may turn to (8) instead and obtain precisely the same region of convergence of the continued fraction (1). *There is no loss in generality in assuming k even in* (2).

In order to identify the curves C_k and determine C_k' it will be convenient to replace z by $1/z'$ and study the corresponding curve E_k in the plane of $z' = 1/z$, and the portion E_k' of E_k which corresponds to C_k'. If, as we now suppose, k is even, say $= 2q$, then H_k is a polynomial of degree q of the form $1 + \sum_1^q A_\nu z^\nu$. We find that

$$\frac{(-p)^{1/2}}{[Z(z)]^{1/2}} = z'^q + A_1 z'^{q-1} + \cdots + A_q.$$

As $Z(z)$ ranges through real values $\leqq 0$, $1/(Z(z))^{1/2}$ ranges through pure imaginary values, z' over E_k, and z over C_k. As $Z(z)$ ranges through real values from $-1/4$ to $-\infty$, $1/(Z(z))^{1/2}$ ranges through pure imaginary values from $-2i$ to $+2i$, z' over E_k', and z over C_k'. Set

$$p^{1/2} = \frac{G}{2} e^{i\phi}, \qquad A_\nu = a_\nu e^{i\alpha_\nu}, \qquad z' = re^{i\theta},$$

$$z'^q + A_1 z'^{q-1} + \cdots + A_q = X + iY,$$

where ϕ is any one of the possible arguments of $p^{1/2}$; G, a_ν are real and positive, and α_ν, θ, r, X, Y are real. We have

$$X = \sum_{\nu=0}^{\eta} a_\nu r^{q-\nu} \cos(\alpha_\nu + \overline{q-\nu}\theta),$$

$$Y = \sum_{\nu=0}^{q} a_\nu r^{q-\nu} \sin(\alpha_\nu + \overline{q-\nu}\theta),$$

($a_0=1$, $\alpha_0=0$). Let t be real. Then E_k is given parametrically by the equations

$$X = t\cos\phi, \qquad Y = t\sin\phi.$$

On eliminating t we find that E_k is the stelloid or Holzmüller hyperbola

$$X\sin\phi - Y\cos\phi = 0, \tag{12}$$

and that E_k' is that part of E_k for which

$$X^2 + Y^2 \leqq G^2. \tag{13}$$

If $q=1$ ($k=2$), E_2 is a straight line, and (13) is the interior of a circle with center on E_2. The curve C_2' is an arc of a circle, and E_4 is a rectangular hyperbola. In case $q=2$, C_4 and C_4' can be determined by the following special method. First determine δ_1, δ_2 by the conditions

$$\begin{aligned} A_1 &\equiv \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 = 2(\delta_1 + \delta_2), \\ A_2 &\equiv \sigma_1\sigma_3 + \sigma_2\sigma_4 = \delta_1^2 + \delta_2^2, \end{aligned} \tag{14}$$

so that

$$\delta_\nu = \frac{A_1 + (-1)^\nu (8A_2 - A_1^2)^{1/2}}{4}, \quad (\nu=1,2), \quad 8\delta_1\delta_2 = A_1^2 - 4A_2.$$

If $\delta_1\delta_2=0$, then $A_1^2 - 4A_2 = 0$. Hence if either A_1 or A_2 is zero, the other is also. If $\delta_1\delta_2 \neq 0$, put

$$\Delta = \frac{\sigma_1\sigma_2\sigma_3\sigma_4}{(\delta_1\delta_2)^2},$$

$$Z_1 = -\frac{(\delta_1\delta_2)^2 z^4}{[1 + 2(\delta_1+\delta_2)z + (\delta_1^2 + \delta_2^2)z^2]^2}.$$

Then $Z = \Delta Z_1$. The function Z_1 is of the form of Z' in (11) (for the case $k=4$), so that

$$Z_1 = -1/(2 + 1/Z_2)^2,$$

where $Z_2 = -\delta_1\delta_2 z^2/(1+\delta_1 z+\delta_2 z)^2$. When $\delta_1\delta_2 = A_1 = A_2 = 0$, it is easy to see that $Z = -\sigma_1\sigma_2\sigma_3\sigma_4 z^4$; and when $\delta_1\delta_2 = 0$, $A_1A_2 \neq 0$, $Z = -\sigma_1\sigma_2\sigma_3\sigma_4 z^4/A_2^2(z+2/A_1)^4$. We find that there are four cases.

CASE 1. $\delta_1\delta_2 = A_1 = A_2 = 0$. The curve C_4' consists of four rays running from z_n to ∞ in the direction from 0 to z_n, where z_n, $(n=1, 2, 3, 4)$, are the four fourth roots of $1/(4\sigma_1\sigma_2\sigma_3\sigma_4)$. In this case $\sigma_1 \neq \sigma_3$ or else $\sigma_2 \neq \sigma_4$, so that the case $k=2$ is not included.

CASE 2. $\delta_1\delta_2 = 0$, $A_1A_2 \neq 0$. In the plane of $z/(z+2/A_1)$ the cut consists of four rays as in Case 1 except that the four fourth roots of $A_2^2/(4\sigma_1\sigma_2\sigma_3\sigma_4)$ are the initial points of the rays. In the plane of z the cut consists of arcs of two circles. Here $\sigma_1 \neq \sigma_3$ or $\sigma_2 \neq \sigma_4$.

CASE 3. $\delta_1\delta_2 \neq 0$, $\delta_1+\delta_2 = A_1/2 \equiv 0$. In the plane of z^2 the cut is an arc of a circle. We may have $\sigma_1 = \sigma_3$, $\sigma_2 = \sigma_4$ if and only if $\sigma_1 = -\sigma_2$. Thus (2) may hold with $k=2$. The cut consists of two rays running to ∞ in this degenerate case.

CASE 4. $\delta_1\delta_2 \neq 0$, $\delta_1+\delta_2 \neq 0$. We may set $\delta_1+\delta_2 = 2\delta$, and apply (11) (for the case $k=2$) to the function $Z_2\delta^2/(\delta_1\delta_2)$. We find that in the plane of $(2+1/(\delta z))^2$ the cut is an arc of a circle. If $\sigma_1 = \sigma_3$, $\sigma_2 = \sigma_4$, $\sigma_1 \neq -\sigma_2$, (2) holds with $k=2$, and the cut is an arc of a circle in the plane of z. If $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4$, (2) holds with $k=1$ and the cut is the ray running from $-1/(4\sigma_1)$ to ∞ in the direction from 0 to $-1/(4\sigma_1)$.

If the σ_n are real and positive it is easy to show that $\delta_1\delta_2 \neq 0$, $\delta_1+\delta_2 \neq 0$, so that Case 4 obtains. In this important case one may show that the cut is a portion of the negative half of the real z axis. The polynomial G_4 is

$$(1 + \sigma_1 z + \sigma_2 z)(1 + \sigma_2 z + \sigma_3 z)(1 + \sigma_3 z + \sigma_4 z)(1 + \sigma_4 z + \sigma_1 z).$$

4. *Examples and Applications.* The following examples have been selected for the purpose of bringing out interesting points which might otherwise be overlooked.

EXAMPLE 1. Let $b_{3n-2} = c_n \neq 0$, $\lim c_n = 0$, $b_{3n+2} = -b_{3n} = a \neq 0$. Here we have, with $k=3$,

$$Z_n^1 = -\frac{z^3a^2c_n}{(1 + c_{n-1}z)(1 + c_nz)},$$

$$Z_n^2 = -\frac{z^3a^2c_n(1 + c_{n-1}z)(1 + c_{n+1}z)}{[1 + (c_n + c_{n-1})z][1 + (c_n + c_{n+1})z]},$$

$$Z_n^3 = -\frac{z^3a^2c_n}{(1+c_nz)(1+c_{n+1}z)}.$$

Since $\lim_{n=\infty} Z_n^m = 0$, $(m=1, 2, 3)$, uniformly over every bounded region, the continued fraction represents a meromorphic function of z by Theorem 2.

EXAMPLE 2. Let $b_{4n+1} = -b_{4n+2} = c_n > 0$, $\lim c_n = 0$, $b_{4n} = -b_{4n-1} = a > 0$. If $b_1 = 1/a_1$, $b_n = 1/(a_{n-1}a_n)$, $(n \geqq 2)$, then $a_{2n+1} > 0$. Since $\sum a_{2n+1}$ diverges, it follows from the work of Hamburger† that the continued fraction converges except upon the real axis. To prove that it represents a meromorphic function‡ it is but necessary to note§ that when $k=4$

$$Z_n^1 = -\frac{z^4a^2c_{n-1}c_{n-2}}{(1-ac_{n-2}z^2)(1-ac_{n-1}z^2)},$$

so that $\lim Z_n^1 = 0$ uniformly over every bounded region, and therefore K_1 represents a meromorphic function.

EXAMPLE 3. If $\limsup|b_n| < g$, $\lim_{n=\infty} b_nb_{n+1} = 0$, we may show that (1) represents a function which is analytic except for poles in the region‖ $|z| \leqq 1/(2g)$. In fact, if $k=2$, $\lim_{n=\infty} Z_n^m = 0$, $(m=1, 2)$, uniformly over this region inasmuch as the polynomials $D_{3,2n+m} = 1+(b_{2n+m+2}+b_{2n+m+3})z$ have no zeros in this region if n is sufficiently large; and $\lim_{n=\infty} b_1^*b_2^* = 0$.

EXAMPLE 4. According to Theorem 1, the real segment $(1, \infty)$ is to be excluded from the region of convergence of the continued fraction (1) if the b's have the values $b_1 = 1/2$, $b_2 = -1/2$, $b_{2n} = -1/(2(1+[n-1]^p/n^p))$, $b_{2n+1} = -1/(2(1+[n+1]^p/n^p))$. One may show that this continued fraction converges or diverges at $z=1$ (a point on this segment) according as p is >1 or $\leqq 1$. In fact, when $z=1$, the nth convergent is

† Mathematische Annalen, vol. 82, pp. 120–187.

‡ See this Bulletin, vol. 39 (1933), pp. 946–952, in which another example is given to show that (1) may represent a meromorphic function when the b_n are real and $b_{2n}b_{2n+1} > 0$, $\limsup|b_n| > 0$. In that example convergence was established except at the poles of the function, whereas here the question of convergence at points on the real axis is not considered.

§ The other Z_n^m, $(m=2, 3, 4)$, are not all so simple in character.

‖ If the condition $\lim b_nb_{n+1} = 0$ is dropped, the same holds in the region $|z| \leqq 1/(4g)$ (see Perron, loc. cit., p. 343).

$$\frac{1}{1^p}+\frac{1}{2^p}+\cdots+\frac{1}{[n/2]^p}$$

when n is even and

$$\frac{1}{1^p}+\frac{1}{2^p}+\cdots+\frac{1}{[(n-1)/2]^p}+\frac{1}{2[(n-1)/2]^p}$$

when n is odd.

As is well known, (1) has a unique corresponding power series $P(z)=\sum c_\nu z^\nu$, $(c_0=1)$, from which (1) may be obtained by a repeated division process. If a given power series $P(z)$ has a corresponding continued fraction of the form (1), it is said to be *semi-normal*.† When convergent, the continued fraction furnishes a method for summing the power series. Let $P'(z)=\sum_{\nu=0}^{\infty} c_{1+\nu}Z^\nu$ be semi-normal with corresponding continued fraction $c_1+K(b_\nu' z/1)$. Then if (1) converges to $f(z)$ when (1) corresponds to $P(z)$, it is important to know conditions under which $c_1+K(b_\nu' z/1)$ converges to the value $(f(z)-1)/z$.

It is well known that if (1) converges to $f(z)$, then when $c_1+K(b_\nu' z/1)$ converges it must have the value $(f(z)-1)/z$. This follows from the fact that the even convergents of $1+c_1 z+zK(b_\nu' z/1)$ are the same as the odd convergents of (1).‡

The numbers b_ν and b_ν' are related as follows.§ Set $a_1=1/b_1$, $a_1'=1/b_1'$, $a_n=1/(b_n a_{n-1})$, $a_n'=1/(b_n' a_{n-1}')$, $(n>1)$. Then if $h_n=a_1+a_3+\cdots+a_{2n+1}$,

$$a_{2n}' = a_{2n+1}/(h_n h_{n-1}), \qquad a_{2n+1}' = a_{2n+2}h_n^2.$$

If (2) holds with k even (say $=2q$), then it is not difficult to show that when the b_n are real and

$$\lim_{n=\infty}(b_{nk+m}/b_{nk+m+1}) = r_m > 0, \qquad (m = 1, 2, 3, \cdots, k), \tag{15}$$

we must also have

$$\lim_{n=\infty}(b'_{nk+m}/b'_{nk+m+1}) = r_m' > 0, \qquad (m = 1, 2, 3, \cdots, k), \tag{16}$$

† Perron, loc. cit., p. 304.

‡ Perron, loc. cit., p. 447.

§ Transactions of this Society, vol. 31 (1929), pp. 102–103.

and

$$\lim_{n=\infty} b'_{nk+m} = \sigma'_m, \qquad (m = 1, 2, 3, \cdots, k). \tag{17}$$

Hence if Theorem 1 is applicable to the continued fraction corresponding to $P(z)$ it is also applicable to the continued fraction corresponding to $P'(z)$ provided $P(z)$ has real coefficients and (15) holds. From (16), (17) it then follows at once that Theorem 1 can be in turn applied to the continued fraction corresponding to $P''(z) = \sum_{\nu=0}^{\infty} c_{2+\nu} z^\nu$, provided the latter is semi-normal, etc. It is easy to conclude from the fact that two successive continued fractions obtained in this way have an infinite number of convergents in common that the function $Z(z)$ of Theorem 1 is the same for all these continued fractions. We shall summarize the result in the following theorem.

THEOREM 3. *If* $P(z) = 1 + c_1 z + c_2 z^2 + \cdots$ *is a semi-normal power series with real coefficients and corresponding continued fraction* (1) *such that for* $k = 2q$ *equations* (2) *and* (15) *hold, and if* $L = c_1 + K(b'_\nu z/1)$ *is the corresponding continued fraction for* $P'(z) = c_1 + c_2 z + c_3 z^2 + \cdots$ *(supposed semi-normal), then* (1) *and* $1 + zL$ *converge to one and the same function* $f(z)$ *over the region* R *described in Theorem* 1. *If* $P^{(n)}(z) = \sum_{\nu=0}^{\infty} c_{n+\nu} z^\nu$ *is semi-normal with corresponding continued fraction* $c_n + K(b_\nu^{(n)} z/1)$, $(n = 1, 2, 3, \cdots, r)$, *then all the continued fractions* $1 + c_1 z + \cdots + c_n z^n + z^n K(b_\nu^{(n)} z/1)$, $(n = 1, 2, 3, \cdots, r)$, *converge over* R *to* $f(z)$.

The continued fractions are precisely the continued fractions of "type 1" of a Padé table,† whose convergents are "stairlike" files of approximants beginning upon the horizontal side of the table. One can show that when $E(z)$, the reciprocal of $P(z)$, and the series $E^{(n)}(z)$ obtained by removing from $E(z)$ the first n terms and the factor z^n, $(n = 1, 2, 3, \cdots)$, are semi-normal, then under the hypothesis of Theorem 3 the continued fractions corresponding to stairlike files of approximants beginning on the vertical side of the table also converge over R to $f(z)$.

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† Perron, loc. cit., pp. 447–448.

ON THE MATRIC EQUATIONS $P(X)=A$ AND $P(A, X)=0$

BY E. T. BROWNE

1. *Introduction.* The matric equation

$$P(X) = A, \tag{1}$$

where $P(\lambda)$ is a polynomial with scalar coefficients and A is a given square matrix of order n, has received a good deal of attention within the past few years. The problem is to find square matrices X satisfying (1). This equation may possess solutions X which are expressible as polynomials in A. On the other hand, there may exist solutions, but none expressible as a polynomial in A; and finally, there are equations of the type (1) which possess no solution at all.*

In 1928 Roth† found necessary and sufficient conditions that there may exist solutions of (1) expressible as polynomials in A, and he found the number of such solutions, in case any exists. He employed the theory of elementary divisors. In this paper Roth gave a bibliography which was quite complete up to that time. In 1931 Franklin attacked the problem through the canonical form, and found not only all solutions X that are expressible as polynomials in A, but also solutions that are not so expressible. Rutherford‡ also employed the canonical form. Still more recently Ingraham§ discussed the problem using the theory of elementary divisors.

Let $F_j(A)$, $(j=0, \cdots, m)$ be known polynomials in A with scalar coefficients, and consider the more general equation

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* Franklin, *Algebraic matric equations*, Journal of Mathematics and Physics, Massachusetts Institute of Technology, vol. 10 (1932), pp. 289–314.

† Roth, *A solution of the matric equation* $P(X)=A$, Transactions of this Society, vol. 30 (1928), pp. 579–596.

‡ Rutherford, *On the canonical form of a rational integral function of a matrix*, Proceedings Edinburgh Mathematical Society, (2), vol. 3 (1932), pp. 135–143.

§ Ingraham, *On the rational solutions of the matrix equation* $P(X)=A$, Journal of Mathematics and Physics, vol. 13 (1934), pp. 46–50.

$$\sum_{j=0}^{m} F_j(A)X^{m-j} = 0, \tag{2}$$

considered by Roth* in a later paper and by Franklin.† This latter equation obviously includes (1) as a special case, and, since it presents little more difficulty than (1), we shall consider (2) first. Our problem then is to find necessary and sufficient conditions that there may exist solutions X of (2) which are expressible as polynomials in A, and to give a simple method for finding such solutions in case any exist. We shall not employ the theory of elementary divisors which leads to a somewhat complicated argument, nor the canonical form, since in actual practise the reduction of a matrix to the canonical form is itself quite tedious. On the other hand, we shall show that the principal *idempotent* and *nilpotent* matrices associated with the matrix A lend themselves quite readily to a solution of the problem.

2. *The Principal Idempotent and Nilpotent Matrices Associated with* A.‡ If λ is a scalar, the equation of degree n,

$$\psi(\lambda) = |\lambda I - A| = 0,$$

is called the *characteristic equation* of A. As is well known, A satisfies its own characteristic equation, but often this is not the equation of *lowest* degree that A satisfies. In fact, if $\theta(\lambda)$ denotes the highest common factor of all the $(n-1)$-rowed minors of $A-\lambda I$, and if we denote by $\phi(\lambda)$ the polynomial $\psi(\lambda)/\theta(\lambda)$, then $\phi(\lambda)=0$ is the equation of lowest degree which A satisfies. We shall hereafter refer to $\phi(\lambda)$ as the *reduced characteristic function* and to the equation $\phi(\lambda)=0$ as the *reduced* or *minimum equation* of A.

Let us suppose that $\phi(\lambda)$ when resolved into linear factors is of the form

$$\phi(\lambda) = \prod_{i=1}^{r} (\lambda - \alpha_i)^{\nu_i}, \qquad (\sum \nu_i = N), \tag{3}$$

* Roth, *On the equation* $P(A,X)=0$ *in matrices*, Transactions of this Society, vol. 35 (1933), pp. 689–708.

† Franklin, loc. cit.

‡ Wedderburn, *Lectures on Matrices*, American Mathematical Society Publications, 1934, pp. 23–29.

where the α's are distinct. For $r>1$, we write

$$h_i(\lambda) = \frac{\phi(\lambda)}{(\lambda - \alpha_i)^{\nu_i}}, \qquad (i = 1, \cdots, r).$$

We can then determine two scalar polynomials $M_i(\lambda)$ and $N_i(\lambda)$ of degrees not exceeding $\nu_i - 1$ and $N - \nu_i - 1$, respectively, such that

$$M_i(\lambda)h_i(\lambda) + N_i(\lambda)(\lambda - \alpha_i)^{\nu_i} \equiv 1.$$

If we write

$$\phi_i(\lambda) = M_i(\lambda)h_i(\lambda), \qquad (i = 1, \cdots, r),$$

and for $\phi_i(A)$ write ϕ_i, then ϕ_i is the *principal idempotent* element of A corresponding to the root α_i. The matrices ϕ_i satisfy the conditions

$$\phi_i^k = \phi_i, \tag{4}$$

for any positive integer k;

$$\phi_i\phi_j = 0, \quad (i \neq j); \qquad \sum_{i=1}^{r} \phi_i = I. \tag{5}$$

Moreover, these ϕ's are linearly independent and none is zero. Let us now denote by η_i the matric polynomial in A defined by the formula

$$\eta_i = \eta_i(A) = (A - \alpha_i I)\phi_i(A), \quad (i=1, 2, \cdots, r).$$

It is easily shown that these matrices η_i satisfy the conditions

$$\eta_i^k \neq 0, \quad (k < \nu_i); \qquad \eta_i^{\nu_i} = 0; \tag{6}$$

$$\eta_i\phi_i = \eta_i = \phi_i\eta_i; \qquad \eta_i\eta_j = 0, \quad (i \neq j); \tag{7}$$

and, moreover,

$$A = \sum_{i=1}^{r} (\alpha_i\phi_i + \eta_i) = \sum \phi_i(\alpha_i + \eta_i).^* \tag{8}$$

The matrix η_i is the *principal nilpotent* element of A corresponding to the root α_i. If $r=1$, so that $\phi(\lambda)$ reduces to $(\lambda-\alpha)^N$, we take $\phi = I$, $\eta = A - \alpha I$, and it will be seen at once that such of

* Here and throughout the remainder of the paper we have designated the scalar matrix $\alpha_i I$ merely by the symbol α_i.

the conditions (4), $\cdots$, (8) as are applicable hold also in this case.

3. LEMMA 1. *If the ϕ's and η's are defined as in the preceding section, and if $f(A)$ is any polynomial in A with scalar coefficients, then*

$$f(A) = \sum_{i=1}^{r} \phi_i f(\alpha_i + \eta_i) = \sum_{i=1}^{r} \phi_i f_i(\eta_i), \tag{9}$$

where the f_i are polynomials determined uniquely by $f(A)$. Conversely, any such expression is equal to a polynomial in A.

For $r>1$, we have from (8), in view of (4), (5), and (7),

$$A^2 = \sum \phi_i(\alpha_i + \eta_i)^2,$$

and, in general, if k is any positive integer

$$A^k = \sum \phi_i(\alpha_i + \eta_i)^k.$$

Hence, if $f(A)$ is a polynomial with scalar coefficients, it follows that

$$f(A) = \sum \phi_i f(\alpha_i + \eta_i).$$

If now $f(\alpha_i+\eta_i)$ be expanded and written as a polynomial $f_i(\eta_i)$ in η_i, we have (9). Moreover, it is clear from the manner in which they arise that the f_i are unique.*

The converse follows at once since each of the ϕ's and η's is a polynomial in A. The lemma also holds obviously for $r=1$.

LEMMA 2. *Let X be a matrix expressible as a polynomial in A, and therefore of the form on the right in* (9). *Necessary and sufficient conditions that X may be a solution of* (2) *are that the f_i satisfy the relations*

$$\sum_{j=0}^{m} F_j(\alpha_i + \eta_i) f_i^{m-j}(\eta_i) = 0, \qquad (i = 1, \cdots, r). \tag{10}$$

For, by Lemma 1, $F_j(A)$ is expressible in the form

$$F_j(A) = \sum_{i=1}^{r} \phi_i F_j(\alpha_i + \eta_i), \qquad (j = 0, \cdots, m).$$

* Otherwise, if we assume that $\sum \phi_i f_i(\eta_i) = \sum \phi_i \psi_i(\eta_i)$, it will follow, just as in the proof of the second part of Lemma 2, that $f_i = \psi_i$, $(i = 1, \cdots, r)$.

Also, if $X=f(A)=\sum_{\tau=0}\phi_\tau f_\tau(\eta_\tau)$, we have as above

$$X^{m-j} = \sum \phi_\tau f_\tau^{m-j}(\eta_\tau).$$

On substituting into (2) and making use of (4) and (5), we have

$$\sum_{i=1}^{r} \phi_i \sum_{j=0}^{m} F_j(\alpha_i + \eta_i) f_i^{m-j}(\eta_i) = 0. \tag{11}$$

The conditions (10) are therefore obviously sufficient. To prove that they are also necessary, multiply (11) through by ϕ_τ, and we obtain

$$\phi_i \sum_{j=0}^{m} F_j(\alpha_i + \eta_i) f_i^{m-j}(\eta_i) = 0, \qquad (i = 1, \cdots, r). \tag{12}$$

Now, by hypothesis, the f_i and all of the F's are polynomials. Hence, since $\eta_i^{\nu_i}=0$, each of the equations (12) reduces to the form

$$\phi_i(a_0 + a_1\eta_i + \cdots + a_{\nu_i-1}\eta_i^{\nu_i-1}) = 0. \tag{13}$$

On multiplying through by η_i and recalling that $\phi_i\eta_i^k=(\phi_i\eta_i)^k$ we obtain

$$a_0(\phi_i\eta_i) + a_1(\phi_i\eta_i)^2 + \cdots + a_{\nu_i-1}(\phi_i\eta_i)^{\nu_i} = 0.$$

Now the minimum equation of $\eta_i=\phi_i\eta_i$ is

$$(\phi_i\eta_i)^{\nu_i} = 0,$$

so that the matrices $\phi_i\eta_i, \cdots, (\phi_i\eta_i)^{\nu_i-1}$ are linearly independent. Hence,

$$a_0 = a_1 = \cdots = a_{\nu_i-2} = 0,$$

and it follows from (13) that we have also $a_{\nu_i-1}=0$. The lemma is therefore established.

4. *The Matric Equation* $P(A, X)=0$. If for brevity we denote the left member of (2) by $P(A, X)$, the conditions (10) can clearly be written in the form

$$P[\alpha_i + \eta_i,\ f_i(\eta_i)] = 0. \tag{10'}$$

Since η_i behaves in all ways precisely as a scalar variable ξ, except that $\eta_i^{\nu_i}=0$, and since a necessary and sufficient condi-

tion that (10′) hold is that the left member be divisible by $\eta_i^{\nu_i}$, we may state the following theorem.

THEOREM 1. *If α_i, $(i=1, \cdots, r)$, are the distinct roots, of multiplicities ν_i, respectively, of the minimum equation of A, necessary and sufficient conditions that a matrix X, expressible as a polynomial in A, satisfy* (2) *are that there exist polynomials $f_i(\xi)$, $(i=1, \cdots, r)$, such that the equations*

$$P[\alpha_i + \xi, f_i(\xi)] = 0, \qquad (i = 1, \cdots, r), \tag{14}$$

possess roots of multiplicities ν_i, respectively, at $\xi=0$. If such polynomials f_i exist, then $X=\sum \phi_i f_i(\eta_i)$ is a solution of (2).

Assuming that the conditions of Theorem 1 are satisfied, let us now proceed to find the polynomials f_i. Since $\eta_i^{\nu_i}=0$, the typical polynomial f_i may be taken in the form

$$f_i(\xi) = x_0 + x_1\xi + \cdots + x_{\nu-1}\xi^{\nu-1}. \tag{15}$$

From (14) we have then as a first necessary condition

$$P(\alpha_i, x_0) = 0. \tag{16}$$

Any root of this equation will serve as x_0. If $\nu_i=1$, f_i is determined. If $\nu_i>1$, we differentiate (14) as to ξ and put $\xi=0$, whence

$$P_{\alpha_i} + x_1 P_{x_0} = 0, \tag{17}$$

where the subscripts denote partial differentiation. If $P_{x_0}\neq 0$, that is, if there exists a simple root of (15) which may be chosen for x_0, then x_1 is uniquely determined. If, however, for every root x_0 of (16), we have $P_{x_0}=0$, it is necessary also that $P_{\alpha_i}=0$ at $\xi=0$. Differentiating (14) a second time and putting $\xi=0$, we have

$$P_{\alpha_i\alpha_i} + 2x_1 P_{\alpha_i x_0} + P_{x_0x_0}x_1^2 + 2P_{x_0}x_2 = 0,$$

and so on.

It will be noticed that the successive coefficients $x_1, x_2, \cdots$, of f_i enter these equations for the first time with the coefficient P_{x_0}. This leads to the Theorem of Franklin.*

THEOREM 2. *A sufficient condition that the equation $P(A, X)=0$ may have a solution X, expressible as a polynomial in A, is that*

* Franklin, loc. cit.

for each multiple root α_i of the minimum equation of A, the equation $P(\alpha_i, x_0) = 0$ have at least one simple root.

5. *Solution of the Matric Equation $P(X) = A$.* In particular, if our matric equation is of the form (1), the equation (14) becomes

$$P[f_i(\xi)] = \alpha_i + \xi. \tag{18}$$

In this case, (17) reduces to

$$x_1 P_{x_0} = 1,$$

so that the condition $P_{x_c} \neq 0$ is not only a *sufficient* condition that x_1 may exist, but a *necessary* one also. Since each simple root of (16) yields a unique polynomial f_i corresponding to it, it follows from the uniqueness feature of Lemma 1 that we are led to the Theorem of Roth.*

THEOREM 3. *Let A be a square matrix whose minimum equation has the s multiple roots $\alpha_1, \cdots, \alpha_s$, and the t simple roots $\alpha_{s+1}, \cdots, \alpha_{s+t}$. The matric equation $P(X) = A$ has a solution for X as a polynomial in A if, and only if, each of the equations*

$$P(x) = \alpha_i, \qquad (i = 1, \cdots, s),$$

has at least one simple root. If these equations have respectively $\mu_1, \cdots, \mu_s$ simple roots and if $\mu_{s+1}, \cdots, \mu_{s+t}$ denote the number of distinct roots of the equations

$$P(x) = \alpha_i, \quad (i = s + 1, \cdots, s + t),$$

the total number of solutions of (1) for X as a polynomial in A is

$$\mu_1 \mu_2 \cdots \mu_{s+t}.$$

COROLLARY. *If the minimum equation of A has all roots distinct, the matric equation $P(X) = A$ is always solvable for X as a polynomial in A. If μ_i denotes the number of distinct roots of the equation $P(x) = \alpha_i$, the total number of such solutions is $\mu_1 \mu_2 \cdots \mu_r$.*

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* Roth, Transactions of this Society, vol. 30 (1928), pp. 579–596.

DISTRIBUTION OF MASS FOR AVERAGES OF NEWTONIAN POTENTIAL FUNCTIONS

BY J. M. THOMPSON

1. *Introduction.* It has been proved that the average of a potential function over a spherical volume and the average of a potential function over a spherical surface are themselves potential functions.* This paper is concerned with the determination of the distribution of mass for these two spherical averages; in addition, the distribution of mass for more general averages is obtained.

2. *Preliminary Theorems.* The problem is solved by means of a theorem on the change of the order of integration of an iterated Stieltjes integral. First it is necessary to state some preliminary theorems. We recall the following elementary theorem.

If $h(Q)$ is continuous in Q and $g(e)$ is a distribution of positive mass, bounded in total amount, and lying on a bounded set F (which may be taken as closed without loss of generality), then, for the integral over the whole of space, w,

$$(1) \qquad \left| \int_w h(Q)dg(e_Q) - \sum_w h(Q_i)g(e_i) \right| < \omega_\delta \alpha,$$

where the summation is extended over all the meshes of a lattice L_δ, of diameter $\leqq \delta$, Q_i is a point of the mesh e_i, ω_δ is the oscillation of $h(Q)$ on a subset of F of diameter $\leqq \delta$, and $\alpha \geqq g(F)$.

This theorem will be applied to the integral

$$\int_w h^N(M, Q)dg(e_Q, P),$$

where $h^N(M, Q)$ is continuous in M, Q, and $g(e, P)$ and F are bounded independently of P, so that ω_δ and α in (1) are independent of M, P.

THEOREM 1. *If $g(e, P)$ is a distribution of positive mass, bounded independently of P, on a set F bounded independently*

* G. C. Evans, *On potentials of positive mass*, Transactions of this Society, vol. 37 (1935), p. 250.

of P, and if $f(e)$ is a distribution of positive mass, bounded in all space, and if $g(e, P)$ is summable with respect to $f(e)$, then

$$G(e) = \int_w g(e, P)df(e_P)$$

is a distribution of positive mass bounded in total amount and lying on the set F.

In this theorem, it is not required that $g(e, P)$ be continuous in P, so that the integral must be interpreted as a generalized or Daniell integral with respect to $f(e)$. But in the theorems which follow, $g(e, P)$ is taken as continuous in P for every mesh of L.

To prove that $G(e)$ is a distribution of positive mass, we must show that $G(e)$ is a non-negative, absolutely additive function of point sets (only Borel measurable point sets are considered).

The first requirement follows immediately from the fact that $g(e, P)$ and $f(e)$ are non-negative for all P and e. In order to exhibit the second condition, let $e = e_1 + e_2 + \cdots$ and $e_n' = e_1 + e_2 + \cdots + e_n$; as $g(e, P)$ is a distribution of positive mass, we have

$$g(e, P) = \lim_{n=\infty} g(e_n', P), \text{ with } g(e_{n+1}', P) \geqq g(e_n', P).$$

Since $g(e, P)$ is, as a function of P, the limit of an increasing sequence of functions, the order of integration and passing to the limit may be interchanged; so that

$$G(e) = \int_w g(e, P)df(e_P) = \lim_{n=\infty} \int_w g(e_n', P)df(e_P)$$
$$= \lim_{n=\infty} [G(e_1) + \cdots + G(e_n)] = G(e_1) + G(e_2) + \cdots.$$

Hence $G(e)$ is a distribution of positive mass. Also $G(e)$ is a distribution of mass lying on F; that is, $G(e) = 0$, if $e \cdot F = 0$, because of the hypothesis that $g(e, P) = 0$, if $e \cdot F = 0$. Finally, $G(e) \leqq$ [upper bound of $g(e, P)$] $\cdot f(w)$, so bounded.

We state the generalizations of two theorems proved by H. E. Bray; the proofs are omitted as they are essentially the same as those given by Bray.* They depend on the inequality (1).

* H. E. Bray, *Elementary properties of the Stieltjes integral*, Annals of Mathematics, vol. 20 (1918–19), pp. 180–185.

THEOREM 2. *If $h^N(M, Q)$ is a continuous function of M and Q, and is bounded in all space, if $g(e, P)$ is a distribution of positive mass, bounded independently of P, on a set F bounded independently of P, and if $g(e, P)$ is continuous in P for every cell e of a net L, then*

$$K(M, P) = \int_w h^N(M, Q)dg(e_Q, P)$$

is continuous in M and P.

THEOREM 3. *If $h^N(M, Q)$ is continuous in M and Q, and bounded in all space, if $g(e, P)$ is a distribution of positive mass, bounded independently of P, on a set F bounded independently of P, and if it is continuous in P for every cell e of a net L, and if $f(e)$ is a distribution of positive mass, bounded in total amount, lying on a bounded set E (which may be taken as closed without loss of generality), then the integrals*

$$\int_w df(e_P) \int_w h^N(M, Q)dg(e_Q, P),$$

and

$$\int_w h^N(M, Q)d\left[\int_w g(e_Q, P)df(e_P)\right]$$

exist and are equal.

We may now state and prove the concluding theorem in this series.

THEOREM 4. *If $g(e, P)$ is a positive distribution of mass, bounded in total amount independently of P, on a set F bounded independently of P, and if it is continuous in P for every cell e of the net L, and if $f(e)$ is a distribution of positive mass, bounded in total amount and lying on the set E, then*

$$\int_w df(e_P) \int_w \frac{1}{MQ} dg(e_Q, P) = \int_w \frac{1}{MQ} d\left[\int_w g(e_Q, P)df(e_P)\right],$$

(or both are $+\infty$), where $\int_w g(e, P)df(e_P)$ is a bounded distribution of positive mass lying on the set F.

If

$$h^N(M, Q) = \frac{1}{MQ}, \text{ if } \frac{1}{MQ} \leqq N,$$

$$= N, \quad \text{if } \frac{1}{MQ} > N,$$

the function $h^N(M,Q)$ is continuous in M and Q.

By Theorem 3, we have

$$\int_w df(e_P) \int_w h^N(M, Q) dg(e_Q, P)$$

$$= \int_w h^N(M, Q) d\left[\int_w g(e_Q, P) df(e_P) \right].$$

Let N become infinite; we have

$$\int_w df(e_P) \lim_{n=\infty} \int_w h^N(M, Q) dg(e_Q, P)$$

$$= \int_w \frac{1}{MQ} d\left[\int_w g(e_Q, P) df(e_P) \right].$$

The interchange of integration and passing to the limit in the left-hand member is justified because the integrand is an increasing function of N, while the definition of the integral of a lower-semi-continuous function with respect to a distribution of positive mass was used in the right-hand member. Applying this definition now to the left-hand member, we have

$$\int_w df(e_P) \int_w \frac{1}{MQ} dg(e_Q, P) = \int_w \frac{1}{MQ} d\left[\int_w g(e_Q, P) df(e_P) \right].$$

This, combined with Theorem 1, establishes the theorem.

3. *Volume Averages.* This theorem just proved enables one to exhibit the distribution of mass for the spherical volume average of a potential function in a form in which it may be evaluated; this is an illustration of the advantage sometimes gained by working with the more general situation.

In the theorems that follow, $u(Q)$ is the potential at Q of the distribution of positive mass, $f(e)$, which is bounded in total amount and lies on a bounded set E.

THEOREM 5. *The average of the potential function $u(Q)$ over a spherical volume $\Gamma(r, M)$, of radius r and center M, is a potential function of a distribution of positive mass with density $(3/(4\pi r^3))f\{\Gamma(r, Q)\}$. In symbols,*

$$
\begin{aligned}
a_u(r, M) &= \frac{3}{4\pi r^3}\int_{\Gamma(r,M)} u(Q)dQ \\
&= \frac{3}{4\pi r^3}\int_w \frac{1}{MQ} f\{\Gamma(r, Q)\}dQ.
\end{aligned} \tag{2}
$$

Let us form this spherical volume average and determine its distribution of mass. We have

$$
\begin{aligned}
a_u(r, M) &= \frac{3}{4\pi r^3}\int_{\Gamma(r,M)} u(Q)dQ \\
&= \frac{3}{4\pi r^3}\int_{\Gamma(r,M)} dQ \int_w \frac{1}{PQ}\, df(e_P).
\end{aligned}
$$

As the integrand is a lower-semi-continuous function, the order of integration may be interchanged, so that

$$
a_u(r, M) = \frac{3}{4\pi r^3}\int_w df(e_P)\int_{\Gamma(r,M)} \frac{1}{PQ}\, dQ.
$$

The inner integral is the potential at P of a sphere of unit density with radius r and center at M, which is equal to the potential at M of the same sphere with center at P. Hence,

$$
\begin{aligned}
a_u(r, M) &= \frac{3}{4\pi r^3}\int_w df(e_P)\int_{\Gamma(r,P)} \frac{1}{MQ}\, dQ \\
&= \frac{3}{4\pi r^3}\int_w df(e_P)\int_w \frac{1}{MQ}\, dg(e_Q, P),
\end{aligned}
$$

where $g(e, P) = m_3\{e \cdot \Gamma(r, P)\}$, and m_3 means the three-dimensional measure of the set indicated. The quantity $g(e, P)$ is evidently continuous in P for every measurable set e.

By means of Theorem 4, this volume average may be expressed in a form which enables the mass distribution to be evaluated,

$$(3) \qquad a_u(r, M) = \frac{3}{4\pi r^3}\int_w \frac{1}{MQ}\, d\Big[\int_w m_3\{e_Q \cdot \Gamma(r, P)\}\, df(e_P)\Big].$$

Consider the integral,

$$v(e) = \int_w m_3\{e \cdot \Gamma(r, P)\}\, df(e_P),$$

where e is an arbitrary bounded set measurable Borel. This function is an absolutely continuous function of e; in fact

$$v(e) \leqq m_3(e) \cdot f(E),$$

and $v(e)$ is completely additive by Theorem 1.

The integrand of $v(e)$ may be expressed as a Lebesgue integral,

$$m_3\{e \cdot \Gamma(r, P)\} = \int_{e \cdot \Gamma(r,P)} 1 dR = \int_e B(R, P) dR,$$

where we may define $B(R, P)$ as 1, for $RP < r$, and 0, for $RP \geqq r$. In this way $B(R, P)$ is the limit of an increasing sequence of continuous functions of R and P, and we have

$$(4) \qquad \begin{aligned} v(e) &= \int_w df(e_P) \int_e B(R, P) dR = \int_e dR \int_w B(R, P) df(e_P) \\ &= \int_e f\{\Gamma(r, R)\}\, dR. \end{aligned}$$

Substituting this result in (3) and making use of the definition of the integral of a lower-semi-continuous function, we have

$$\begin{aligned} a_u(r, M) &= \frac{3}{4\pi r^3}\int_w \frac{1}{MQ}\, d\Big[\int_{e_Q} f\{\Gamma(r, R)\}\, dR\Big] \\ &= \frac{3}{4\pi r^3} \lim_{N=\infty} \int_w h^N(M, Q) d\Big[\int_{e_Q} f\{\Gamma(r, R)\}\, dR\Big]. \end{aligned}$$

As $h^N(M, Q)$ is bounded and continuous, and $v(e)$ is absolutely continuous, we may change the Stieltjes integral to a Lebesgue integral. This gives

$$a_u(r, M) = \frac{3}{4\pi r^3} \lim_{N=\infty} \int_w h^N(M, Q) f\{\Gamma(r, Q)\}\, dQ$$

$$= \frac{3}{4\pi r^3} \int_w \frac{1}{MQ} f\{\Gamma(r, Q)\} dQ,$$

and the proof is complete.

It should be pointed out that the work of this section holds for the average over any three-dimensional open set. Let $s(O)$ be an open set (therefore of positive spatial measure), $s(M)$ the set obtained by displacement of $s(O)$ as a rigid body, without rotation, so that O falls on M, and let $s'(Q)$ be the reflection of the set $s(M)$ through the midpoint of the line MQ.

THEOREM 6. *The average of the potential function $u(Q)$ over the set $s(M)$ is a potential function of a distribution of positive mass with density $(1/[m_3 s(M)])\, f\{s'(Q)\}$. In symbols,*

$$a_u\{s(M)\} = \frac{1}{m_3\{s(M)\}} \int_{s(M)} u(Q) dQ = \frac{1}{m_3\{s(M)\}} \int_w \frac{1}{MQ} f\{s'(Q)\} dQ.$$

The construction of the set $s'(P)$ gives the potential at P of the set $s(M)$, of unit density, equal to the potential at M of the set $s'(P)$, of unit density; hence all the transformations made in this section on the spherical volume average are valid for the average over the set $s(M)$.

4. *Spherical Surface Average.* By means of Theorem 4, we may also determine the mass function for the average of a potential function over a spherical surface.

THEOREM 7. *The average of the potential function $u(Q)$ over the spherical surface $C(r, M)$, of radius r and center M, is a potential function of the distribution of positive mass,*

$$\frac{1}{4\pi r^2} \int_m m_2\{e \cdot C(r, P)\} df(e_P).$$

In symbols,

$$A_u(r, M) = \frac{1}{4\pi r^2} \int_{C(r,M)} u(Q) dQ = \frac{1}{4\pi r^2} \int_w \frac{1}{MQ}\, d\left[\int_w m_2\{e_Q \cdot C(r, P)\} df(e_P)\right].$$

Treating the spherical surface average in the same manner as we have treated the volume average, we obtain

$$\begin{aligned} A_u(r, M) &= \frac{1}{4\pi r^2}\int_{C(r,M)} u(Q)dQ = \frac{1}{4\pi r^2}\int_{C(r,M)} dQ \int_w \frac{1}{QP} df(e_P) \\ &= \frac{1}{4\pi r^2}\int_w df(e_P) \int_{C(r,M)} \frac{1}{QP} dQ \\ &= \frac{1}{4\pi r^2}\int_w df(e_P) \int_{C(r,P)} \frac{1}{QM} dQ \\ &= \frac{1}{4\pi r^2}\int_w df(e_P) \int_w \frac{1}{MQ} dg(e_Q, P), \end{aligned}$$

where $g(e, P) = m_2\{e \cdot C(r, P)\}$. For a given P, this function is additive and bounded for cells e of a three-dimensional lattice, and hence can be extended by definition uniquely to all sets spatially measurable Borel.

As $g(e, P)$ is a continuous function of P for every cell e of a net L, Theorem 4 applies, and we may thereby express this average in a form that exhibits its mass function in terms of $C(r, P)$ and $f(e)$,

$$(5) \qquad A_u(r, M) = \frac{1}{4\pi r^2}\int_w \frac{1}{MQ}\, d\left[\int_w m_2\{e_Q \cdot C(r, P)\} df(e_P)\right].$$

This result requires no restriction on $f(e)$ other than those we have already stated. However, we shall state also a special case of Theorem 7.

THEOREM 8. *If $f\{\Gamma(r, Q)\}$ is differentiable with respect to r, and, for a fixed neighborhood of the given value of r, $\partial f\{\Gamma(n, Q)\}/\partial r$ is bounded independently of the point Q, then the average of the potential function $u(Q)$ over the spherical surface $C(r, M)$ is a potential function of a distribution of positive mass with density $(1/(4\pi r^2))\partial f\{\Gamma(r, Q)\}/\partial r$. In symbols,*

$$(6) \qquad A_u(r, M) = \frac{1}{4\pi r^2}\int_w \frac{1}{MQ} \frac{\partial f\{\Gamma(r, Q)\}}{\partial r} dQ.$$

We have, for a rectangular cell e,

$$(7) \qquad m_2\{e \cdot C(r, P)\} = \lim_{i=\infty} \frac{m_3\{e \cdot [\Gamma(r_i, P) - \Gamma(r, P)]\}}{r_i - r},$$

where $r < r_{i+1} < r_i$ and $\lim_{i=\infty} r_i = r$.

Using the results given in (4), we have the following equality,

$$\int_w \frac{m_3\{e\cdot[\Gamma(r_i, P) - \Gamma(r, P)]\}}{r_i - r}\, df(e_P) = \int_e \frac{f\{\Gamma(r_i, Q)\} - f\{\Gamma(r, Q)\}}{r_i - r}\, dQ.$$

The integrand of the left-hand member belongs to a sequence of measurable, uniformly bounded functions, as a function of P, whose limit exists when i becomes infinite; so we let i become infinite and interchange the order of integration and pass to the limit for the left-hand member. The same considerations hold for the integrand of the right-hand member as a function of Q. Using (13), we have

$$\int_w m_2\{e\cdot C(r, P)\}\, df(e_P) = \int_e \frac{\partial f\{\Gamma(r, Q)\}}{\partial r}\, dQ.$$

The quantity $\partial f\{\Gamma(r, Q)\}/\partial r$ is non-negative. Hence we may substitute this last equation in (5) and change the Stieltjes integral into a Lebesgue integral as we did above for the volume average. Thus we have established the theorem.

THE UNIVERSITY OF CALIFORNIA

ERRATUM

In my paper entitled *On the summability of a certain class of series of Jacobi polynomials* (this Bulletin, vol. 41 (1935), pp. 541–549), the following change should be made; it conforms with the last proofs that I had seen.

Page 544, 8th line from the bottom: read $S_{n,\lambda}^{(k)}$ instead of $S_{n,k}^{(k)}$.

A. P. COWGILL

OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

NEW YORK CITY, October 26, 1935.

Abstracts must be in the hands of Associate Secretary J. R. Kline, Low Memorial Library, Columbia University, New York City, not later than September 28. Abstracts received by the Secretaries by September 6 will appear in the September issue of this BULLETIN. By invitation of the Program Committee, Professor J. L. Walsh will deliver an address on *Interpolation by rational functions.*

LEXINGTON, KENTUCKY, November 29–30, 1935.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than November 1. All these will appear in the November issue of this BULLETIN. By invitation of the Program Committee, addresses will be delivered by Professor L. L. Dines on *Convex domains and linear inequalities*, and by Professor C. G. Latimer on *The arithmetic of generalized quaternions.*

CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, November 30, 1935.

Abstracts must be in the hands of Associate Secretary T. M. Putnam, University of California, Berkeley, California, not later than November 2. All these will appear in the November issue of this BULLETIN.

ST. LOUIS, MISSOURI, ANNUAL MEETING, December 31, 1935–January 3, 1936.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than November 20, 1935. Abstracts received by the secretaries by November 2 will appear in the November issue of this BULLETIN. This meeting will be held in conjunction with meetings of the A.A.A.S. and the Mathematical Association of America. By invitation of the Program Committee, addresses will be delivered by Professor Szegö on *Some recent investigations concerning the sections of trigonometric and related series*, and by Professor Synge on *Tensorial methods in dynamics.* On Tuesday afternoon, at a joint session with the Mathematical Association and Section A of the A.A.A.S., Professor Carmichael will deliver his retiring address as Chairman of Section A, on *Linear differential equations of infinite order.* The twelfth Josiah Willard Gibbs Lecture will be delivered by Professor Vannevar Bush on the topic *Mechanical analysis.*

R. G. D. RICHARDSON, *Secretary of the Society.*

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

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The initiation fees and the annual dues of members of the Society (see this BULLETIN, p. 322, May, 1930; and the List of Officers and Members, September, 1934, p. 66), are payable to the Treasurer of the Society, Professor G. W. Mullins, Low Memorial Library, Columbia University, New York City.

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